Methods in Quantitative Risk Management

M.Tech. Dissertation

Submitted in partial fulfillment of the requirements

for the degree of

Master of Technology

by

Saket Sathe

Roll No: 03307914

under the guidance of

Prof. Uday B. Desai

Department of Electrical Engineering,

Indian Institute of Technology, Bombay.

Mumbai - 400 076.

Acknowledgment

I sincerely express my gratitude towards my guide **Prof. U. B. Desai** and also **Prof. R. K. Lagu** for their constant encouragement and invaluable guidance during the course of this project.

Saket K. Sathe.

Department of Electrical Engineering,

IIT Bombay.

Abstract

We have shown that NSE (National Stock Exchange) is inefficient as compared to NYSE (New York Stock Exchange) [1]. In particular we have carried out weak-form market efficiency analysis. We employ methods from *information theory* and *chaos theory*. *Risk-Metrics*® methodology for estimating *Value-at-Risk* assumes that log-returns of stock indices follow a normal distribution. Our analysis has revealed that the log-returns of the NSE indices are more leptokurtic. Thus, unlike the Gaussian distribution, we use *fat-tailed* distributions, like, Generalized Hyperbolic distribution (GHD) and Normal-Inverse Gaussian (NIG) distribution for computing and comparing the revised *Value-at-Risk* estimates for the NSE. For comparison we have also carried out weak-form market efficiency analysis for the NYSE (New York Stock Exchange). Here, it was found that the log-returns of the NYSE indices could be better approximated by normal distribution. Nevertheless, in the univariate case GHD and NIG distribution provides a much superior model. Next, we turn our attention to portfolio *VaR* estimation. We use copula methods to compute and compare portfolio *VaR* estimates. We observed that copulas were insufficient in capturing extreme co-movements which existed in our analyzed portfolios. As a result we use the multivariate counterpart of the Generalized Hyperbolic distributions. Here too we find that the Generalized Hyperbolic distributions provide a much realistic model for *VaR*.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

"..the biggest problem we now have with the whole evolution of risk is the fat-tailed problem, which is really creating very large conceptual difficulties. Because as we all know, the assumption of normality enables us to drop off the huge amount of complexity in our equations. Because once you start putting in non-normality assumptions, which is unfortunately what characterizes the real world, then these issues become extremely difficult."

– Alan Greenspan, Chairman, Federal Reserve Board (1997).

Market efficiency is the ability of a market to provide accurate signals for forming *fair* prices. Fama [2] summarizes this idea in his classic survey by writing: "A market in which prices always *fully reflect* available information is called *efficient*." In an efficient market the expected value for gains is always *zero*. Arbitrage opportunities are ruled out in an efficient market. Thus, market efficiency acts as a measure of justification for the application of theoretical concepts to actively traded markets.

We show that National Stock Exchange (NSE) is *weak-form* inefficient compared to New York Stock Exchange (NYSE) [1]. Our analysis has revealed that the log-returns of the NSE indices are more leptokurtic. Thus, unlike the Gaussian distribution, we use *fat-tailed* distributions, like, Generalized Hyperbolic distribution (GHD) and Normal-Inverse Gaussian (NIG) distribution for computing and comparing the revised *Value-at-Risk* estimates for the NSE.

Initial development of Hyperbolic distributions was done by Barndorff-Nielson [3]. He applied the Hyperbolic subclass to fit grain size of sand subjected to continuous wind

flow. Later he [4] generalized the concepts of Hyperbolic distributions to the Generalized Hyperbolic distributions (GHD). But, Eberlin and Keller [5] were the first to use these distributions in finance. Recently, Prause [6] used the GHD to fit financial data, using German stocks and American indices, extending the work of Eberlin and Keller [5]. Prause [6] has also developed a program to estimate the GHD parameters, but these programs are not freely available.

It is well known that the Gaussianity assumption on the stock indices return series might throw *Value-at-Risk (VaR)* estimates astray. Thus, we examine the implication of using *fat-tailed* distributions like the GHD and the Normal-Inverse Gaussian (NIG) distribution. We find that for the NSE indices they provide a much more reliable estimate of *VaR* than the ones obtained under the Gaussianity assumption.

It is worthwhile to check the validity of our approach by applying it to other prominent stock markets in the world. We have chosen the world's leading stock market, namely, NYSE (New York Stock Exchange) for this purpose. After comparing the NYSE with the NSE we find the behavior of NYSE to be quite different. The log-returns of the NYSE indices exhibit low leptokurticity and thus Gaussianity works quite well. Nevertheless, for accurate *VaR* estimates we need to exploit the GHD and the NIG distributions.

We then turn our attention to portfolio *VaR* estimation. Here we have used copula methods to calibrate a Gaussian and Student's t copula to real market portfolios and compare the *VaR* estimates obtained from both the copulas. We observed that although the Student's t copula bear heavier tails than the Gaussian copula it is insufficient to capture all the extreme co-movements inherent in the portfolios analyzed. As a result we use the multivariate Generalized Hyperbolic distributions to obtain more realistic estimates of portfolio *VaR*. These subclass of distributions are known to have heavier tails than the Student's t distribution, but the estimation of these distributions for high-dimensional data is challenging. We overcome this problem by using an Expectation-Maximization type algorithm [7] for estimation of a *fifteen* dimensional Generalized Hyperbolic distribution.

1.1 Organization of the dissertation

The remainder of the dissertation is organized as follows: In Chapter 2 we examine the summary statistics of the NSE and the NYSE stock indices returns series. Chapter 3 comprises of various tests for weak-form market efficiency. In Chapter 4, we use *fat-tailed* distributions, like, the Generalized Hyperbolic distribution and the Normal Inverse Gaussian distribution, to compute and compare VaR estimates for the NSE and NYSE stock indices returns series. We estimate the Generalized Hyperbolic distributions using the Maximum Likelihood approach in Section (4.2). In Section (4.4), we validate the improved accuracy of our approach through backtesting. Then in Chapter 5, we introduce copula functions and describe algorithms for calibration and random variate generation from a joint distribution using copulas. We calibrate various copulas to real market portfolios and compare portfolio VaR estimates obtained from various copulas. Lastly, in Chapter 6 we use multivariate Generalized Hyperbolic distributions to obtain significantly realistic portfolio VaR estimates. Also, we describe and implement an EM type algorithm for calibration of multivariate Generalized Hyperbolic distributions.

Chapter 2

National Stock Exchange (NSE) and New York Stock Exchange (NYSE) Data

The analyzed data consist of closing value of six National Stock Exchange (NSE) indices, namely, S&P CNX Nifty (NIFTY), CNX Nifty Junior (NIFJUN), S&P CNX 500 (CNX500), CNX Midcap 200 (CNXMID), CNX IT (CNXIT)[1], and BANK Nifty (NIFBNK). We have also analyzed two NYSE (New York Stock Exchange) indices, namely, Dow Jones Industrial (DOWJI) and S&P 500 (S&P500). The daily rates of return were computed as,

$$r_t = 100 \times \ln\left(\frac{S_t}{S_{t-1}}\right), \qquad (2.1)$$

where r_t is the rate of return and S_t is the stock market index closing price at moment t. The above computation assumes multiplicative model of asset pricing dynamics [8].

The summary statistics for the daily returns [2] of the chosen indices are shown in Table (2.1). It is a well known fact that daily stock market returns defy the assumption of normality [9]. They exhibit a week evidence of *skewness* and strong evidence of *kurtosis*. All the stock indices return series [3] display varying order of leptokurticity. NYSE stock indices returns have significantly less kurtosis. The CNXBNK index exhibits high kurtosis

---

[1] The base value of this index was revised from 1000 to 100 w.e.f 28th May 2004. This has been taken into consideration.

[2] Hereafter, we use 'returns' and 'log-returns' synonymously.

[3] Henceforth, collectively addressed as, the NSE and the NYSE stock indices returns series.

Table 2.1: Summary statistics of logarithmic returns

Exchange	NSE						NYSE	
Index	NIFTY	NIFJUN	CNX500	CNXMID	CNXIT	CNXBNK	DOWJI	S&P500
From	Jan-1996	Feb-1996	Jun-1999	Jun-1999	Jan-1996	Jan-2000	Jan-1996	Jan-1996
To	Apr-2005	Apr-2005	Apr-2005	Apr-2005	Apr-2005	Apr-2005	Apr-2005	Apr-2005
No. of samples	2300	2300	1480	1480	2300	1300	2328	2328
Mean	0.0280	0.0530	0.0505	0.0938	0.1480	0.0810	0.0300	0.0273
Median	0.0631	0.0967	0.1667	0.2333	0.1444	0.0500	0.0402	0.0421
SD	1.6311	1.9042	1.7030	1.6860	2.9476	1.9064	1.1541	1.1865
Variance	2.6606	3.6262	2.9003	2.8425	8.6886	3.6345	1.3319	1.4079
Minimum	-13.0539	-13.0174	-12.8847	-11.6405	-22.1257	-15.1380	-7.4549	-7.1127
Maximum	9.9339	8.2100	7.6945	6.7929	14.5567	11.4014	6.1547	5.5744
Skewness	-0.3190	-0.5607	-0.8194	-0.9258	-0.3777	-0.6841	-0.2243	-0.0895
Kurtosis	7.8782	6.8697	8.1356	7.6885	7.2714	10.2301	5.2216	5.7521
Jarque-Bera Test	2313.10	1550.95	1784.77	1560.57	1797.20	2920.36	1329.60	735.15

compared to other indices. The CNXIT index has the highest variance. The high variance of CNXIT can be attributed to high volatility of Indian IT (Information Technology) stocks. The null hypothesis of normality is clearly rejected for all the indices using the Jarque-Bera test at 1% significance level.

We have processed the NSE and the NYSE stock indices returns series for possible linear and/or nonlinear dependence [1]. The word 'dependence' is obviously connected to the fact that dependence means *predictability*. The motivation behind examining nonlinear dependence comes from the fact that the returns series violate the assumption of normality which is a basis for the linear dependence study. Recall, stock market inefficiency implies *dependence* in the returns series. In an efficient market the returns are *i.i.d.* Our analysis has revealed that the NSE is inefficient as compared to NYSE. Also, NSE indices are far more leptokurtic than the NYSE indices. Thus, to get a realistic estimate of *VaR*, we use *heavy-tailed* distributions, like the Generalized Hyperbolic subclass of distributions.

Chapter 3

Market Efficiency of NSE and NYSE Indices

3.1 Serial Correlation

Serial (auto) correlation is one of the oldest metric for dependence [1]. The sample serial correlation coefficients are estimated as follows.

Let $\{x_t\}_{t=1}^{t=T}$, where $T \in \mathbb{Z}^+$, be a time series. Then the serial correlation of $\{x_t\}_{t=1}^{t=T}$ at lag h is defined as,

$$\rho_x(h) = \frac{\sum_{t=1}^{T-h}(x_{t+h} - \bar{x})(x_t - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})(x_t - \bar{x})}, \tag{3.1}$$

where $h = 1, 2, \ldots, H$; $H < T$ and $\bar{x} = \frac{1}{T}\sum_{t=1}^{t=T} x_t$.

It is often required, by looking at the first few serial correlation coefficients, whether the null hypothesis that the original time series being random can be accepted. Box *et al.* [11] suggested a *portmanteau* test for randomness. The test statistic,

$$Q_x(H) = T(T+2)\sum_{j=1}^{H} \frac{\rho_x^2(j)}{T-j}, \tag{3.2}$$

is approximately $\chi^2_{1-\alpha;H}$ distributed, where α is the significance level and H is the degrees of freedom. The null hypothesis of the original time series being random is rejected if,

$$Q_x(H) > \chi^2_{1-\alpha;H}. \tag{3.3}$$

---

[1]See [10] for the history of serial correlation.

Table 3.1: Ljung-Box test statistic and serial correlation coefficients for the returns series.

Exchange	NSE						NYSE	
Index	NIFTY	NIFJUN	CNX500	CNXMID	CNXIT	CNXBNK	DOWJI	S&P500
$\rho_r(1)$	0.0598	0.1415	0.1412	0.1473	0.1399	0.1145	-0.0088	-0.0145
$\rho_r(2)$	-0.0469	-0.0304	-0.0371	-0.0179	0.0390	-0.0858	-0.0305	-0.0294
$\rho_r(3)$	0.0186	0.0273	0.0282	0.0232	0.0237	0.0063	-0.0110	-0.0288
$\rho_r(4)$	0.0380	0.0131	0.0472	0.0673	0.0214	0.0318	0.0204	0.0092
$\rho_r(5)$	0.0019	0.0157	0.0073	0.0235	0.0006	-0.0221	-0.0362	-0.0417
$\rho_r(6)$	-0.0606	-0.0314	-0.0230	-0.0184	-0.0164	-0.0533	0.0006	-0.0099
$\rho_r(7)$	-0.0073	-0.0048	0.0110	-0.0313	-0.0226	0.0190	-0.0346	-0.0424
$\rho_r(8)$	0.0013	0.0168	0.0306	0.0180	0.0677	-0.0118	0.0167	0.0102
$\rho_r(9)$	0.0355	0.0622	0.0217	0.0220	0.0432	0.0269	0.0058	0.0037
$\rho_r(10)$	0.0611	0.1168	0.0684	0.0937	0.0674	0.0958	0.0276	0.0167
$\rho_r(20)$	-0.0236	-0.0314	-0.0337	-0.0366	-0.0450	-0.0134	-0.0173	-0.0260
$\rho_r(30)$	-0.0308	-0.0334	-0.0638	-0.0632	-0.0259	-0.0131	-0.0165	-0.0092
$\rho_r(40)$	0.0576	0.0086	0.0184	0.0071	0.0223	0.0178	-0.0325	-0.0276
$\rho_r(50)$	-0.0141	-0.0314	-0.0132	0.0050	0.0307	-0.0453	-0.0114	-0.0107
$\rho_r(60)$	0.0108	0.0154	0.0074	-0.0020	-0.0047	0.0234	0.0059	0.0035
$\rho_r(70)$	-0.0200	0.0088	-0.0180	0.0004	-0.0184	0.0020	0.0170	0.0169
$\rho_r(80)$	0.0102	-0.0057	-0.0287	-0.0310	-0.0027	-0.0224	0.0145	0.0041
$\rho_r(90)$	-0.0250	0.0090	0.0078	0.0114	-0.0084	-0.0464	0.0046	0.0103
$\rho_r(100)$	-0.0232	-0.0032	-0.0367	-0.0183	-0.0223	-0.0014	0.0343	0.0194
$Q_r(50)$	95.5603	147.8036	95.7744	117.8784	152.9821	101.9468	70.8255	69.8925
$Q_r(100)$	143.741	197.9624	146.9926	164.7046	202.9214	148.6747	132.5288	127.4192

The Ljung-Box statistic and the serial correlation coefficients for the NSE and the NYSE stock indices returns series, $\{r_t\}_{t=1}^{t=T}$, are presented in Table (3.1). Ljung-Box statistic rejects the null hypothesis of the NSE stock indices returns series being random at $\alpha = 1\%$ and the NYSE stock indices returns series being random at $\alpha = 5\%$. Consquently, small amount of *serial dependence* is found. Many stock indices around the world show small amounts of serial dependence [12, 13, 14]. It is interesting to note that compared to other indices, $\rho_r(1)$ of NIFTY is nearly half. $\rho_r(1)$ of DOWJI and S&P500 is much smaller compared to the NSE indices. This clearly suggests a presence of *short run memory* and *low* linear dependence in the DOWJI and S&P500 indices. The second lag serial correlation coefficient is significantly negative in seven of eight stock indices returns series which supports the so called *mean-reversion* behavior of stock market returns [13].

3.2   Global Self-Correlation

Granger *et al.* [15] have discussed ideal properties of a good dependence measure. *Mutual information* [16, 17], although a powerful tool for quantifying dependencies does not satisfy some of these properties. However, a function of mutual information can be shown to satisfy properties of a good dependence measure.

Global correlation [12, 18] is one such function of mutual information, that satisfies all the properties of a good dependence measure [15]. *Global correlation* is derived from mutual information as,

$$\lambda(X,Y) = \sqrt{1 - e^{-2M(X,Y)}}, \tag{3.4}$$

where

$$M(X,Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) log \left(\frac{p(x,y)}{p(x)p(y)} \right), \tag{3.5}$$

$p(x) = Pr\{X = x\}, x \in \mathcal{X}$, the probability mass function *(p.m.f)* of the random variable X, $p(x,y)$ is the joint *p.m.f* between random variables X and Y, $p(y) = Pr\{Y = y\}, x \in \mathcal{Y}$, the *p.m.f* of the random variable Y, and $M(X,Y)$ is the mutual information between random variables X and Y [2]. It is easily seen that *global correlation* varies between 0 and 1.

Global correlation has a few noteworthy properties [12]:

1. $\lambda(X,Y) = 0$, iff X contains *no* information about Y, which implies Y cannot be predicted from X;

2. $\lambda(X,Y) = 1$, iff X contains *all* the information about Y, which implies Y can be completely predicted from X. This is a limiting case of determinism.

Accordingly, we define the *global self-correlation* for $\{x_t\}_{t=1}^{t=T}$ as,

$$\lambda_x(h) = \sqrt{1 - e^{-2\langle M(x_t, x_{t-h}) \rangle}}, \tag{3.6}$$

where $\langle \cdot \rangle$ denotes time average. This gives the predictability of the time series at time t from its value at time $t - h$ for all t's.

---

[2]For the logarithm to the base 2 the mutual information is measured in *bits*; logarithm to base 10 mutual information measure is *dits*; logarithm to the base e the mutual information measure is *nats*. Here we are using logarithm to the base e.

Table 3.2: Global self-correlation coefficients for the returns series.

Exchange	NSE						NYSE	
Index	NIFTY	NIFJUN	CNX500	CNXMID	CNXIT	CNXBNK	DOWJI	S&P500
$\lambda_r(1)$	0.136966	0.242791	0.234785	0.248164	0.303801	0.119931	0.079716	0.082278
$\lambda_r(2)$	0.080348	0.154920	0.159956	0.196079	0.220364	0.025745	0.108039	0.124123
$\lambda_r(3)$	0.049617	0.148508	0.065755	0.066476	0.185760	0.038903	0.091987	0.147606
$\lambda_r(4)$	0.037716	0.078165	0.060414	0.132959	0.133374	0.042782	0.050309	0.070825
$\lambda_r(5)$	0.056959	0.164509	0.109783	0.097078	0.163656	0.028939	0.106275	0.126756
$\lambda_r(6)$	0.060163	0.095471	0.166774	0.089960	0.189485	0.042016	0.101233	0.055854
$\lambda_r(7)$	0.067711	0.035604	0.116168	0.071027	0.148343	0.067618	0.066523	0.073025
$\lambda_r(8)$	0.050972	0.055767	0.096801	0.089744	0.143865	0.053198	0.065177	0.036942
$\lambda_r(9)$	0.033485	0.092591	0.161507	0.059638	0.143111	0.045566	0.050409	0.036239
$\lambda_r(10)$	0.029581	0.038969	0.120167	0.063084	0.155632	0.029949	0.063596	0.143205
$\lambda_r(15)$	0.042322	0.060336	0.070484	0.056786	0.128213	0.055516	0.062818	0.032247
$\lambda_r(20)$	0.034441	0.071093	0.083308	0.101470	0.125272	0.063895	0.043245	0.049660
$\lambda_r(25)$	0.044649	0.021322	0.115321	0.087162	0.141170	0.113567	0.043510	0.079219
$\lambda_r(30)$	0.053889	0.073169	0.025478	0.034236	0.108535	0.063366	0.023989	0.046652

The global self-correlation for the NSE and the NYSE stock indices returns series are given in Table (3.2). All the indices, except the NYSE indices (columns 8 and 9), have significant global self-correlation coefficient at lag one. The CNXIT index displays an average 0.15 global self-correlation coefficient persisting for as long as twenty lags. This indicates strong presence of nonlinear dependence in the CNXIT index. On the other hand, CNXBNK index reveal small global self-correlation. The small global self-correlation coefficients of the NYSE indices indicate negligible presence of nonlinear dependence.

The above empirical analysis suggests that the NYSE is more efficient as compared to the NSE.

3.3 BDS Test: Testing for $i.i.d$

We use a test originally proposed by Brock, Dechert and Scheinkman, henceforth referred to as the BDS test. BDS test bears the null hypothesis of the data being originated from an $i.i.d$ process [19]. It involves computing the following statistic.

Let $\{x_t\}_{t=1}^{t=T}$ be a sequence of observations that are $i.i.d$. Form N-dimensional vectors $x_t^N = (x_t, x_{t+1}, \ldots, x_{t+N-1})$. These are called N-histories and N is known as the

embedding dimension. Compute,

$$C_N(l,T) = \frac{2}{T_N(T_N - 1)} \sum_{i<j} I_l(x_i^N, x_j^N), \tag{3.7}$$

where $T_N = T - N + 1$. $I_l(x, y)$ is an indicator function that equals 1 if $||x - y|| < l$, and 0 otherwise, where $||\cdot||$ is a norm, and $C_N(l, T)$ is known as the *sample correlation integral*. Brock, Dechert, and Scheinkman show that under the null hypothesis of $\{x_t\}$ being *i.i.d*,

$$C_N(l,T) \to C_1(l,T)^N \text{ with probability one, as } T \to \infty, \tag{3.8}$$

where $C_1(l,T)^N$ is the *true* correlation integral for any fixed N and l, and the BDS statistic,

$$w_N(l,T) = \sqrt{T}\left[\frac{C_N(l,T) - C_1(l,T)^N}{\sigma_N(l,T)}\right], \tag{3.9}$$

where

$$\sigma_N^2(l,T) = 4\left[K(l,T)^N + 2\sum_{J=1}^{N-1} K(l,T)^{N-j} C_1(l,T)^{2j} + (N-1)^2 C_1(l,T)^{2N}\right.$$
$$\left. - N^2 K(l,T) C_1(l,T)^{2N-2}\right],$$

and

$$K(l,T) = \frac{6}{T_N(T_N-1)(T_N-2)} \sum_{t<s<r} I_l(x_t, x_s) I_l(x_s, x_r),$$

has a standard normal distribution, $N(0,1)$. An intuitive explanation of the BDS test can be given. The correlation integral $C_N(l,T)$ is an estimate of the probability that any two N-histories, $x_t^N = (x_t, x_{t+1}, \ldots, x_{t+N-1})$ and $x_s^N = (x_s, x_{s+1}, \ldots, x_{s+N-1})$, are within l of each other. Thus, if the x_t's are independent, then, for $|t - s| > N$,

$$C_N(l,T) \to \prod_{i=0}^{N-1} \Pr\{|x_{t+i} - x_{s+i}| < l\}, \text{ as } T \to \infty.$$

BDS statistic also provides elementary information about the type of dependence exhibited by the data. The probability of any two N-histories, $x_t^N = (x_t, x_{t+1}, \ldots, x_{t+N-1})$ and $x_s^N = (x_s, x_{s+1}, \ldots, x_{s+N-1})$, being *close* is higher than the Nth power of the probability of any two points, x_t and x_s, being together. In other words, some form of *clustering* is occurring frequently in an N-dimensional space. Thus some regular patterns appear in the stock market returns.

The BDS test was performed on the residuals of the NSE and the NYSE stock indices returns series, $\{r_t\}_{t=1}^{t=T}$. Residuals were obtained by fitting an *Autoregressive Process of*

14

order p, AR(p), to the returns series. The order of *AR(p)* model was chosen using the *AIC (Akaike Information Criteria)*. This careful procedure is followed because the BDS test has high power against linear dependence and we want to make sure that linear effects are completely eliminated. Another important issue in performing the BDS test is the choice of l. A good choice of l could be in terms of the standard deviation of the data, that is, $l = 1$ means that it is one standard deviation of data. We use a similar procedure for choosing l.

Table (B.1), Table (B.2) and Table (B.3) (see Appendix B) show the BDS statistic for the NSE and the NYSE stock indices returns series for embedding dimensions, $N = 1, 2, \ldots, 16$ and for $l/\sigma = 0.5, 0.75, 1, 1.25, 1.5$. Clearly, the null hypothesis of the returns series being *i.i.d* is rejected in all cases at 5% significance level.

Thus, although the NYSE is more efficient compared to the NSE it cannot be classified as a *truly* efficient market.

Chapter 4

Value-at-Risk

Regulatory guidelines require that a firm (bank) should set aside capital to insure against extreme portfolio losses. These guidelines are aimed at reducing the probability of bankruptcy which generally occur when a firm's risk exposure is not in proportion with the firm's capital. Currently, portfolio risk is measured in terms of *Value-at-Risk (VaR)*.

$RiskMetrics^{\circledR}$ [20] is a popular methodology for estimating *VaR*. It assumes that the stock market returns follow a Gaussian distribution. On the other hand, the above analysis clearly showed that the NSE stock indices are *fat-tailed* (leptokurtic) and deviate significantly from the assumption of normality.

Value-at-Risk (VaR) is the maximum amount a market position can loose for a given time horizon at a given confidence level. In mathematical terms, a portfolio's *VaR* is defined as the negative of the inverse image of a α quantile under the probability distribution function of the daily returns series. Then, *Value-at-Risk*, $VaR(\alpha)$ is,

$$VaR(\alpha) = -F^{-1}(\alpha), \tag{4.1}$$

where $F(x)$ is the probability distribution function of the daily returns series. This α should not be confused with the significance level. For example, if a portfolio's daily 1% *VaR*, $VaR(0.01)$, is US \$10,000,000 then 1% of the times a portfolio's daily loss is in excess of US \$10,000,000.

In this section we investigate *VaR* estimates when the returns are fitted with Normal-Inverse Gaussian (NIG) distribution, Generalized Hyperbolic distribution (GHD) and Hyperbolic (HYP) distribution. We show that, for the daily returns series, the GHD and the NIG distribution give a much more reliable estimate of *VaR* as compared to the

ones obtained using the $RiskMetrics^{\circledR}$ [20] approach which is based on the assumption of normality.

4.1 Generalized Hyperbolic Distributions

The one-dimensional Generalized Hyperbolic distribution of a random variable x is defined by the following Lebesgue density,

$$
\begin{aligned}
gh(x; \lambda, \alpha, \beta, \delta, \mu) &= a(\lambda, \alpha, \beta, \delta)(\delta^2 + (x - \mu)^2)^{(\lambda - \frac{1}{2})/2} \times \\
&\quad K_{\lambda - 1/2}\left(\alpha\sqrt{\delta^2 + (x - \mu)^2}\right) \times exp\left(\beta(x - \mu)\right), \\
a(\lambda, \alpha, \beta, \delta) &= \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda - 1/2}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)},
\end{aligned}
\tag{4.2}
$$

where K_λ is a modified Bessel function of the third kind [1] and $x \in \mathbb{R}$. The domain of variation of the parameters is $\mu \in \mathbb{R}$,

$$
\begin{aligned}
\delta &\geq 0, |\beta| < \alpha \text{ if } \lambda > 0 , \\
\delta &> 0, |\beta| < \alpha \text{ if } \lambda = 0 , \\
\delta &> 0, |\beta| \leq \alpha \text{ if } \lambda < 0 .
\end{aligned}
$$

Different scale and location invariant parameterizations of the Generalized Hyperbolic distribution have been proposed in the literature.

$$
\begin{aligned}
\zeta &= \delta\sqrt{\alpha^2 - \beta^2}, \quad \varrho = \beta/\alpha & \ldots \text{2nd parametrization,} & \tag{4.3} \\
\xi &= (1 + \zeta)^{-1/2}, \quad \chi = \xi\varrho & \ldots \text{3nd parametrization,} & \tag{4.4} \\
\overline{\alpha} &= \alpha\delta, \quad \overline{\beta} = \beta\delta & \ldots \text{4nd parametrization.} & \tag{4.5}
\end{aligned}
$$

If the distribution is symmetric then $\beta = \overline{\beta} = \varrho = \chi = 0$ holds. The parameters μ and δ describe the location and the scale, whereas β describes the skewness. Increasing ξ or decreasing ζ (or $\overline{\alpha}$) reflect an increase in the kurtosis. The log-density of the Gaussian distribution is a parabola while the log-density of GHD is a hyperbola, thus the name–GHD. Mean and variance of the Generalized Hyperbolic distribution are given as follows,

$$
E[x] = \mu + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)},
\tag{4.6}
$$

$$
Var[x] = \delta^2\left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} + \frac{\beta^2}{\alpha^2 - \beta^2}\left[\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \left(\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}\right)^2\right]\right),
\tag{4.7}
$$

---

[1]For more details about Bessel functions, see [21].

where $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$. It can be observed that the formulae for mean and variance are less complicated in the symmetric case ($\beta = 0$).

Many distributions can be obtained as a subclass or limiting distribution of GHD. Using Bessel function with index $\mathbb{N} + \frac{1}{2}$ we can obtain a simpler subclass of distributions. When $\lambda = 1$ in (4.2) we get the Hyperbolic distribution with Lebesgue density,

$$hyp(x; \alpha, \beta, \delta, \mu) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\delta\alpha K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} exp\left(-\alpha\sqrt{\delta^2 + (x - \mu)^2} + \beta(x - \mu)\right), \quad (4.8)$$

where $x, \mu \in \mathbb{R}$, $0 \leq \delta$ and $|\beta| < \alpha$. This distribution is not closed under convolution.

The Normal-Inverse Gaussian distribution can be obtained by setting $\lambda = -1/2$ in (4.2), the resulting Lebesgue density is,

$$nig(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} exp\left(-\alpha\sqrt{\delta^2 + (x - \mu)^2} + \beta(x - \mu)\right) \frac{K_1(\alpha\sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}},$$
$$(4.9)$$

where $x, \mu \in \mathbb{R}$, $0 \leq \delta$ and $0 \leq |\beta| \leq \alpha$. This distribution is a mixture of Generalized Inverse Gaussian distribution and a normal distribution. The NIG distribution can approximate most hyperbolic distributions very closely. A detailed discussion on GHD, NIG distribution and Hyperbolic distribution can be found in [6, 22, 23, 24, 25].

4.2 Maximum-Likelihood Estimation

Here we describe the maximum likelihood estimation algorithm for parameter estimation of the GHD, the NIG distribution and the Hyperbolic distribution. Let, $\{x_t\}_{t=1}^{t=T}$ be independent observations. We maximize the log-likelihood function (obtained from (4.2)),

$$
\begin{aligned}
\mathcal{L}(x_t; \hat{\Theta}) =\ & T \cdot \log\left(a(\lambda, \alpha, \beta, \delta)\right) + \left(\frac{\lambda}{2} - \frac{1}{4}\right) \sum_{t=1}^{T} \log(\delta^2 + (x_t - \mu)^2) \\
& + \sum_{t=1}^{T} \log\left(K_{\lambda-1/2}(\alpha\sqrt{\delta^2 + (x_t - \mu)^2})\right) + \beta \sum_{t=1}^{T} (x_t - \mu), \quad (4.10)
\end{aligned}
$$

with respect to $\hat{\Theta} = \{\lambda, \alpha, \beta, \delta, \mu\}$, where a is the same as in (4.2). In the above equation we use $\lambda = 1$ for estimating the Hyperbolic distribution and $\lambda = -1/2$ for estimating Normal-Inverse Gaussian distribution. Direct maximization of (4.10) is not an easy task because the log-likelihood function is flat and involves the computation of a Bessel function. Considering this, we use the Nelder-Mead simplex search method [26] for maximization of the log-likelihood function. This is a direct search method that does not use

18

numerical or analytic gradients. Convergence is accelerated if the initial conditions for the maximization are chosen properly. We choose a symmetric GHD with reasonable kurtosis as an initial distribution i.e. $\beta = 0$ and $\xi \approx 0.7$ [6]. The termination tolerance for the algorithm was set to 1×10^{-10} in all the cases. Karlis [23] proposed an EM (Expectation-Maximization) type algorithm for the estimation of NIG distribution. We implemented this algorithm using $Matlab^{\circledR}$. This implementation is freely available at [27].

Table 4.1: Maximum-likelihood estimates of Generalized Hyperbolic distribution (GHD), Normal-Inverse Gaussian (NIG) distribution and Hyperbolic distribution (HYP) for the returns series.

Exchange		NSE						NYSE	
Index		NIFTY	NIFJUN	CNX500	CNXMID	CNXIT	CNXBNK	DOWJI	S&P500
λ	GHD	-2.2148	-0.6269	-0.7665	-0.9011	0.2863	-2.0384	0.1783	0.7901
	NIG	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
	HYP	1	1	1	1	1	1	1	1
α	GHD	0.1587	0.4071	0.4757	0.4969	0.3681	0.0229	-0.0373	-0.0379
	NIG	0.6823	0.4376	0.5432	0.5977	0.2470	0.5001	0.9589	0.9235
	HYP	1.0309	0.8045	0.9300	0.9825	0.4929	0.8183	1.4347	1.3739
β	GHD	-0.0336	-0.0524	-0.1210	-0.1667	0.0039	-0.0194	0.0796	0.0806
	NIG	-0.0394	-0.0525	-0.1228	-0.1702	0.0028	-0.0073	-0.0408	-0.0396
	HYP	-0.0470	-0.0573	-0.1418	-0.1940	0.0024	0.0079	-0.0425	-0.0449
μ	GHD	0.1203	0.2417	0.3823	0.5280	0.1143	0.1512	1.8924	1.4537
	NIG	0.1337	0.2415	0.3860	0.5356	0.1228	0.1064	0.0833	0.0828
	HYP	0.1511	0.2493	0.4254	0.5828	0.1280	0.0540	0.0847	0.0894
δ	GHD	2.5841	1.6440	1.6045	1.7124	1.2560	2.7423	-2.2945	-0.9212
	NIG	1.7293	1.5585	1.4460	1.4869	2.1919	1.7491	1.2524	1.2921
	HYP	0.9170	0.5023	0.4785	0.5969	0.2503	0.6315	0.6598	0.6534
LogLH	GHD	4354.36	4559.82	2759.55	2744.36	5547.73	2577.92	3520.32	3603.99
	NIG	4240.34	4559.87	2759.63	2744.55	5549.72	2580.58	3522.62	3604.05
	HYP	4244.04	4566.85	2762.99	2748.48	5550.22	2584.22	3526.09	3605.38
Iterations	GHD	489	372	495	448	375	421	619	452
	NIG	163	230	215	196	163	163	163	179
	HYP	243	302	263	278	241	201	232	253

The maximum-likelihood estimates of the GHD, NIG distribution and Hyperbolic distribution for the NSE and the NYSE stock indices returns series , $\{r_t\}_{t=1}^{t=T}$, are shown in Table (4.1). GH distribution consistently takes more number of iterations than the NIG distribution or the Hyperbolic distribution.

4.3   Goodness of fit

The degree to which the estimated distributions agree with the empirical distributions is measured by their *goodness-of-fit*. For the estimated distributions it is tested using

the following distance measures. Let $F_{emp}(x)$ and $F_{est}(x)$ be the empirical cumulative distribution and estimated cumulative distribution of the random variable x. Then,

1. *Kolmogorov distance* is defined as,

$$KS\left(F_{emp}(x), F_{est}(x)\right) = \max_{x \in \mathbb{R}}\{|F_{emp}(x) - F_{est}(x)|\}, \qquad (4.11)$$

   Kolmogorov distance is more suitable for continuous distributions.

2. *Kuiper distance* is defined as,

$$KP\left(F_{emp}(x), F_{est}(x)\right) = \max_{x \in \mathbb{R}}\{F_{emp}(x) - F_{est}(x)\} + \max_{x \in \mathbb{R}}\{F_{est}(x) - F_{emp}(x)\}, \quad (4.12)$$

   The difference between Kolmogorov and Kuiper distance is that the later considers the upper differences different from lower differences.

3. *Anderson-Darling distance* is given as,

$$AD\left(F_{emp}(x), F_{est}(x)\right) = \max_{x \in \mathbb{R}} \frac{|F_{emp}(x) - F_{est}(x)|}{\sqrt{F_{est}(x)(1 - F_{est}(x))}}. \qquad (4.13)$$

   Unlike the Kolmogorov distance or the Kuiper distance the Anderson-Darling distance concentrates on the tail distances. As risk measures concentrate on the tails of a distribution this turns our to be an useful measure of fit.

4. L^1-*norm and* L^2-*norm* are defined as,

$$L^1\left(F_{emp}(x), F_{est}(x)\right) = \sum_{x \in \mathbb{R}} |F_{emp}(x) - F_{est}(x)|. \qquad (4.14)$$

   Similarly, L^2-norm can be defined as,

$$L^2\left(F_{emp}(x), F_{est}(x)\right) = \sqrt{\sum_{x \in \mathbb{R}} |F_{emp}(x) - F_{est}(x)|^2}. \qquad (4.15)$$

The *goodness-of-fit* of the estimated distributions using the different distance measures discussed above are presented in Table (4.2), Figure (B.1), Figure (B.2), and Figure (B.3). The Gaussian (normal) distribution consistently fits poorly in the tails for the NSE stock indices returns series, while the GHD, the NIG distribution and the Hyperbolic distribution perform much better. It is worth noting that for the NYSE stock indices returns series the Gaussian distribution gives an acceptable fit. In the case of the NYSE the

Gaussianity assumption works well since the NYSE is an actively traded market as compared to the NSE. Nevertheless, even for NYSE, to get a superior estimate for *VaR* and a better *goodness-of-fit*, one needs to use GHD and NIG distribution. This is evident from Table (4.2). By looking at the L^1-norm and the L^2-norms of these distributions we conclude that the GHD fits better than the rest of the distributions in all cases.

Table 4.2: Distance measures of *goodness-of-fit*.

Exchange	NSE						NYSE	
Index	NIFTY	NIFJUN	CNX500	CNXMID	CNXIT	CNXBNK	DOWJI	S&P500
AD								
Normal	0.598751	0.754941	0.589640	0.565503	2.085716	0.833305	0.213846	0.231414
GHD	0.029940	0.041698	0.031939	0.047949	0.038601	0.067149	0.028132	0.036031
NIG	0.032993	0.041615	0.031418	0.047875	0.046666	0.064021	0.033752	0.034558
HYP	0.051320	0.054820	0.043855	0.052134	0.048569	0.061711	0.049643	0.044718
KS								
Normal	0.145276	0.190900	0.170810	0.176982	0.265054	0.197851	0.074509	0.083998
GHD	0.011787	0.010786	0.008354	0.013100	0.010699	0.025817	0.006813	0.011186
NIG	0.011067	0.010778	0.009024	0.013078	0.009970	0.020626	0.008632	0.010592
HYP	0.012546	0.016751	0.010758	0.014355	0.015148	0.018565	0.008788	0.008157
KP								
Normal	0.290305	0.381752	0.334185	0.339985	0.521036	0.373350	0.144215	0.164867
GHD	0.021177	0.019444	0.016039	0.024891	0.017012	0.047368	0.012962	0.017492
NIG	0.020848	0.019782	0.016528	0.025768	0.019101	0.039986	0.016725	0.016861
HYP	0.021734	0.027433	0.020702	0.027754	0.024083	0.031232	0.016769	0.015881
L^1								
Normal	52.053103	67.442669	38.709830	38.030243	92.204890	37.997178	26.442897	28.168282
GHD	1.977051	1.815753	0.985753	1.232032	1.827352	2.657172	1.137844	1.390578
NIG	2.199025	1.900010	0.972307	1.277783	2.335740	2.503077	1.201200	1.385982
HYP	2.290896	2.863220	1.327094	1.653280	2.124473	2.396131	1.398870	1.589823
L^2								
Normal	2.416935	3.137705	2.243010	2.211157	4.285222	2.337327	1.218919	1.309632
GHD	0.100812	0.092678	0.062841	0.081175	0.095589	0.184719	0.056151	0.074577
NIG	0.107421	0.096268	0.061880	0.084265	0.113017	0.167346	0.065105	0.073390
HYP	0.113761	0.153812	0.083465	0.106695	0.118695	0.157473	0.073995	0.079631

4.4   Backtesting

The accuracy of a putative *VaR* model can be gauged using a variety of tests. Almost all the tests used for this purpose focus on a particular transformation of the reported *VaR* and realized profit or loss.

Let us denote the event that the loss on a portfolio exceeds its reported $VaR(\alpha)$ during

a fixed interval of time (one day) by a *hit function*. It is defined as,

$$\eta_t(\alpha) = \begin{cases} 1 & \text{if } x_t \leq -VaR(\alpha), \\ 0 & \text{if } x_t > -VaR(\alpha). \end{cases} \tag{4.16}$$

Assuming that the *VaR* violations are independent, Christoffersen [28] points out that the problem of determining the accuracy of a *VaR* model can be reduced to determining whether the *hit function sequence*, $\{\eta_t(\alpha)\}_{t=1}^{t=T}$, satisfies the *unconditional coverage property*. It states that a *VaR* model is accurate if $\Pr(\eta_t(\alpha) = 1)$ is exactly equal to α in the sequence $\{\eta_t(\alpha)\}_{t=1}^{t=T}$. Here,

$$\hat{\alpha} = \Pr(\eta_t(\alpha) = 1) = \frac{\sum_{t=1}^{t=T} \eta_t(\alpha)}{T}. \tag{4.17}$$

Based on the above equation, Kupiec [29] proposed a *proportion of failures* or *POF* test statistic. *POF* examines the number of times a financial institution's *VaR* is violated over a given span of time. If the number of violations differs considerably from $\alpha \times 100\%$ of the sample, then the accuracy of the underlying risk model becomes questionable. Considering a sample of T observations, the *POF* test statistic is defined as,

$$POF = 2\log\left(\left(\frac{1-\hat{\alpha}}{1-\alpha}\right)^{T-\sum_{t=1}^{t=T}\eta_t(\alpha)} \cdot \left(\frac{\hat{\alpha}}{\alpha}\right)^{\sum_{t=1}^{t=T}\eta_t(\alpha)}\right). \tag{4.18}$$

Table 4.3: Estimates of $\hat{\alpha} \times 100\%$ and *POF* test statistic for the *VaR* of a portfolio with value of one currency unit (1-USD or 1-INR) and one-day time horizon.

Exchange	NSE						NYSE	
Index	NIFTY	NIFJUN	CNX500	CNXMID	CNXIT	CNXBNK	DOWJI	S&P500
$\hat{\alpha} \times 100\%$								
Normal	1.70	2.74	2.16	2.30	2.04	1.92	1.55	1.33
GHD	1.04	1.26	0.88	0.95	1.09	1.08	0.90	1.03
NIG	1.04	1.26	0.88	0.95	1.04	1.08	0.86	1.03
HYP	1.13	1.43	1.08	1.22	1.22	1.38	0.95	1.12
POF								
Normal	9.3161	47.6695	15.1537	18.4106	19.4525	8.8086	6.0169	2.3422
GHD	0.0442	1.4603	0.2306	0.0445	0.1726	0.0758	0.2332	0.0223
NIG	0.0442	1.4603	0.2306	0.0445	0.0442	0.0758	0.4902	0.0223
HYP	0.3819	3.8709	0.0958	0.6538	1.0312	1.7347	0.0724	0.3093

Upon examining closely, the statistic reveals that its value will be zero if the proportion of *VaR* violations $\hat{\alpha} \times 100\%$ is exactly equal to $\alpha \times 100\%$. The *POF* test statistic grows as the proportion of *VaR* voilations differs from $\alpha \times 100\%$ indicating that the proposed *VaR*

measure either systematically understates or overstates the portfolio's underlying level of risk.

The estimates of $\hat{\alpha} \times 100\%$ and the POF test statistic for the VaR of a portfolio with a value of one currency unit (1-USD or 1-INR) and one-day time horizon are shown in Table (4.3). As seen from Table (4.3), the GH distribution and the NIG distribution perform twice as better than the Gaussian distribution in satisfying the *unconditional converge property* evaluated for backtesting at 1% VaR. Though the Gaussian distribution provides acceptable results for the NYSE indices, here too the GHD and NIG distribution exhibit a superior performance than the Gaussian distribution.

Thus, by using GHD and NIG distribution we can obtain a realistic VaR estimate. Until now we have presented models for VaR estimation for returns on a single asset. In Chapter 5 we introduce copula functions. They can be effectively used to model risk embedded in a portfolio of assets [30]. Further, in Chapter 6 we discuss multivariate counterparts of Generalized Hyperbolic distributions (MGHD). We then compare the VaR estimates obtained from copulas and MGHD.

Chapter 5

Copulas

Copulas are functions that join or *couple* multivariate distribution functions to their one-dimensional marginal distribution functions. The study of copulas is quite recent. Copulas are of interest to statisticians for two main reasons: Firstly, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions, sometimes with a view to simulation [31]. The word *copula* is a Latin noun which means "a link, tie, bond", and is used in grammar and logic to describe "that part of a proposition which connects the subject and predicate".

Recently, profound applications of copulas for risk management and derivative pricing have been found [30, 32, 33]. Obviously, this makes the study of copulas interesting. One of the primary application of copulas is in simulation and Monte Carlo studies. Copulas provide a simple way for obtaining random variates from a multivariate distribution.

Copulas allow a bottom-up approach to multivariate dependence modeling. This is typically useful because our knowledge of univariate margins is well understood. These univariate margins can then be used to model multivariate dependence using copulas. An example of their application is in credit risk management where the default probabilities of individual obliger is known while their joint default probabilities are difficult to estimate.

In this chapter we shall define copulas and study their important properties. We shall describe an elementary algorithm for generating random variates, especially from a multivariate distribution, using copulas. We also discuss classification of copulas and three broad methods for calibrating them to real multivariate data. Later, we provide numerical results for *VaR* estimates on a popular equity mutual fund portfolio (Kotak 30) and compare the results obtained from various copulas.

5.1 Preliminaries

Let $\mathbb{R}$ denote $(-\infty, +\infty)$, $\overline{\mathbb{R}}$ denote $[-\infty, +\infty]$, and $\overline{\mathbb{R}}^2$ denote the extended real plane $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

Definition 5.1 (H-volume) *Let S_1 and S_2 be nonempty subsets of $\overline{\mathbb{R}}$, and let H be a function such that its domain, $DomH = S_1 \times S_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle all of whose vertices are in $DomH$. Then the H-volume of B is given by,*

$$V_H(B) \triangleq H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1). \tag{5.1}$$

Definition 5.2 (2-increasing real function) *A 2-place real function H is a function whose domain, $DomH$, is a subset of $\overline{\mathbb{R}}^2$ and whose range, $RanH$, is a subset of $\mathbb{R}$. A 2-place real function H is 2-increasing if $V_H(B) \geq 0$ for all rectangles B whose vertices lie in $DomH$.*

Definition 5.3 (grounded functions) *We say that a function H from $S_1 \times S_2$ into $\mathbb{R}$ is grounded if $H(x, a_2) = 0 = H(a_1, y)$ for all (x, y) in $S_1 \times S_2$.*

Definition 5.4 (subcopula) *A two-dimensional subcopula is a function C' with the following properties:*

1. *$DomC' = S_1 \times S_2$, where S_1 and S_2 are subsets of $\mathbb{I}$ containing 0 and 1;*

2. *C' is grounded and 2-increasing;*

3. *For every u in S_1 and every v in S_2,*

$$C'(u, 1) = u \text{ and } C'(1, v) = v. \tag{5.2}$$

Note that for every (u, v) in $DomC'$, $0 \leq C'(u, v) \leq 1$, so that $RanC'$ is also a subset of $\mathbb{I}$.

Definition 5.5 (copula) *The unit square $\mathbb{I}^2$ is the product of $\mathbb{I} \times \mathbb{I}$ where $\mathbb{I} = [0, 1]$. A two-dimensional copula is a 2-subcopula C whose domain is $\mathbb{I}^2$.*

Equivalently, a copula is a function C from $\mathbb{I}^2$ to $\mathbb{I}$ with the following properties:

1. For every u and v in $\mathbb{I}$,

$$C(u, 0) = 0 = C(0, v) \tag{5.3}$$

   and

$$C(u, 1) = u \text{ and } C(1, v) = v; \tag{5.4}$$

2. For every u_1, u_2, v_1, v_2 in $\mathbb{I}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0. \tag{5.5}$$

Theorem 5.1 (Fréchet-Hoeffding bounds) *Let C' be a subcopula. Then for every (u, v) in DomC',*

$$W(u, v) \leq C'(u, v) \leq M(u, v) \tag{5.6}$$

where $W(u, v) = max(u + v - 1, 0)$ and $M(u, v) = min(u, v)$. W and M are refereed to as the Fréchet-Hoeffding lower and upper bounds respectively.

Proof: Refer [31]

Theorem 5.2 *Let C be a copula. For any v in $\mathbb{I}$, the partial derivative $\partial C/\partial u$ exists for almost all u, and for such v and u,*

$$0 \leq \frac{\partial}{\partial u} C(u, v) \leq 1. \tag{5.7}$$

Similarly, for any u in $\mathbb{I}$, the partial derivative $\partial C/\partial v$ exists for almost all v, and for such u and v,

$$0 \leq \frac{\partial}{\partial v} C(u, v) \leq 1. \tag{5.8}$$

Furthermore, the functions $u \mapsto \partial C(u, v)/\partial v$ and $v \mapsto \partial C(u, v)/\partial v$ are defined and non-decreasing almost everywhere on $\mathbb{I}$.

Proof: Refer [31]

5.2 Sklar's Theorem

Sklar's theorem is central to the theory of copulas, and is the foundation of many applications of copula theory. It provides a relationship between multivariate distribution and their univariate counterparts.

Theorem 5.3 (Sklar's Theorem) *Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all x, y in $\overline{\mathbb{R}}$,*

$$H(x, y) = C(F(x), G(y)). \tag{5.9}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on RanF $\times$ RanG. Conversely, if C is a copula and F and G are distribution functions, then the function H is a joint distribution function with margins F and G.

Proof: Refer [31]

5.2.1 Important properties of Copulas

Definition 5.6 (radial symmetry) *A random variable X is radially symmetric about $\boldsymbol{a}$ if $X - \boldsymbol{a} \triangleq \boldsymbol{a} - X$.*

Definition 5.7 (exchangeability) *A random variable X is exchangeable if,*

$$(X_1, \ldots, X_d) \triangleq (X_{\prod(1)}, \ldots, X_{\prod(d)}), \tag{5.10}$$

for any permutation $(\prod(1), \ldots, \prod(d))$ of $(1, \ldots, d)$.

Definition 5.8 (comonotonicity) *The random variables $X_1, \ldots, X_d$ are said to be comonotonic if they admit as copula the Fréchet upper bound, $M(\cdot)$.*

Proposition 5.1 *The random variables $X_1, \ldots, X_d$ are monotonic if and only if*

$$(X_1, \ldots, X_d) \triangleq (\nu_1^\uparrow(Z), \ldots, \nu_d^\uparrow(Z)), \tag{5.11}$$

for some random variable Z and increasing functions $(\nu_1^\uparrow(\cdot), \ldots, \nu_d^\uparrow(\cdot))$.

Proof: Refer [7] pp. 199.

Definition 5.9 (countermonotonicity) *The random variables X_1 and X_2 are countermonotonic if they have a as copula the Fréchet lower bound, $W(\cdot)$.*

Proposition 5.2 *The random variables X_1 and X_2 are countermonotonic if and only if*

$$(X_1, X_2) \triangleq (\nu_1^\uparrow(Z), \nu_2^\downarrow(Z)), \tag{5.12}$$

with $\nu_1^\uparrow(\cdot)$ increasing and $\nu_2^\downarrow(\cdot)$ decreasing or vice versa.

Proof: Refer [7] pp. 200.

Almost all the definitions given above can be extended to the multivariate case. However, here we shall only state Sklar's theorem for multivariate copulas. For the remaining definitions we refer the reader to Nelsen [31].

Theorem 5.4 (Sklar's theoremn in d-dimensions) *Let H be a d-dimensional distribution function with margins $F_1, F_2, \ldots, F_d$. Then there exits an d-copula C such that for all $x_1, x_2, \ldots, x_d$ in $\overline{\mathbb{R}}^d$,*

$$H(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)). \tag{5.13}$$

If $F_1, F_2, \ldots, F_d$ are continuous, then C is unique; otherwise, C is uniquely determined on $\mathrm{Ran}F_1 \times \mathrm{Ran}F_2 \cdots \times \mathrm{Ran}F_d$. Conversely, if C is an d-copula and $F_1, F_2, \ldots, F_d$ are distribution functions, then the function H defined by (5.13) is an d-dimensional distribution function with margins $F_1, F_2, \ldots, F_d$.

5.3 Classification of Copulas

There are many classes of elliptical copulas. Elliptical copulas are broadly classified as Gaussian, Student's t, skewed Student's t, and Archemedian copulas. Clayton, Gumbel, and Frank copulas belong to the family of Archemedian copulas. Elliptical copulas provide elaborate applications in modeling dependence in random variables and have numerous applications. We start this section by defining elliptical distributions.

Definition 5.10 (elliptical distribution) *A random variable X has an elliptical distribution if*

$$X \triangleq \mu + AY, \tag{5.14}$$

where $Y \sim S_k(\psi)$, $A \in \mathbb{R}^{d \times k}$, and $\mu \in \mathbb{R}^d$ are a matrix and vector of constants. Here the notation $S_k(\psi)$ indicates that ψ is a characteristic generator of a k-dimensional spherical distribution. Thus elliptical distributions are obtained by multivariate affine transformations of spherical distributions. The matrix $A'A$ is denoted by Σ and we denote elliptical distributions as,

$$X \sim \mathfrak{E}_d(\mu, \Sigma, \psi), \tag{5.15}$$

where the vector μ is known as a location vector, Σ as a dispersion matrix and ψ as the characteristic generator of the distribution.

Definition 5.11 (characterstic generator) *The characteristic generator ψ of a spherically distributed random variable X is defined as*

$$\psi(t't) = \psi(t_1^2 + \cdots + t_d^2) = \mathbb{E}(e^{it'X}), \qquad \text{for all } t \in \mathbb{R}^d. \tag{5.16}$$

5.3.1 Gaussian Copula

Definition 5.12 *Let R be a symmetric, positive definite matrix with $diag(R) = 1$ and let Φ_R the standardized multivariate normal distribution with correlation matrix R. Then the multivariate Gaussian copula is defined as,*

$$C(u_1, u_2, \ldots, u_n; R) \triangleq \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n)), \tag{5.17}$$

where $\Phi^{-1}(u)$ denotes the inverse of the normal cumulative distribution function. The density function of the Gaussian copula is given as,

$$c(u_1, u_2, \ldots, u_n; R) \triangleq \frac{1}{|R|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\zeta'(R^{-1} - I)\zeta\right]. \tag{5.18}$$

where $\zeta = (\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n))'$.

There are stable and computationally efficient estimators for R. Thus, a Gaussian copula and multivariate Gaussian distribution can be used interchangeably.

5.3.2 Student's t copula

The Student's t copula gives us models that emphasize tail dependence. These copulas are heavier in tails than the Gaussian copula. But, the methods for calibrating Student's t copulas are computationally more expensive than the ones required to calibrate a Gaussian copula.

Definition 5.13 *Let R be a symmetric, positive definite matrix with $diag(R) = 1$ and let $T_{R,\nu}$ the standardized multivariate Student's t distribution with correlation matrix R and ν degrees of freedom. Then the multivariate Student's t copula is defined as follows,*

$$C(u_1, u_2, \ldots, u_n; R, \nu) \triangleq T_{R,\nu}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_n)), \tag{5.19}$$

where $t_\nu^{-1}(u)$ denotes the inverse of the Student's t cumulative distribution function. The associated Student's t copula density is obtained by applying equation,

$$c(u_1, u_2, \ldots, u_n; R, \nu) \triangleq |R|^{\frac{1}{2}} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})} \left[\frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}\right]^n \frac{(1 + \frac{\zeta' R^{-1} \zeta}{\nu})^{-\frac{\nu+n}{2}}}{\prod_{i=1}^{n}(1 + \frac{\zeta_i^2}{\nu})^{-\frac{\nu+1}{2}}}, \tag{5.20}$$

where $\zeta = (t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_n))'$.

Gaussian and Student's t copula are called *implicit* copulas as they are implied by well known multivariate probability distributions and do not themselves have simple closed forms.

5.3.3 Archemedian Copulas

Archemedian copulas initially appeared in the study of probabilistic metric spaces, particularly in the construction of probabilistic version of triangular inequality [31]. Recently, Archemedian copulas have found numerous applications in credit risk management.

Archemedian copulas have the following general from,

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)), \tag{5.21}$$

for an appropriately defined *inverse*, φ^{-1}. This *inverse* is defined as [31],

Definition 5.14 *Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, \infty]$ such that $\varphi(1) = 0$. The* pseudo-inverse *of φ is the function φ^{-1} with $\mathrm{Dom}\varphi^{-1} = [0, \infty]$ and $\mathrm{Ran}\varphi^{-1} = \mathbb{I}$ given by,*

$$y = \begin{cases} \varphi^{-1}(t), & 0 \le t \le \varphi(0), \\ 0, & \varphi(0) \le t \le \infty. \end{cases} \tag{5.22}$$

Also, φ^{-1} is continuous and nonincreasing on $[0, \infty]$, and strictly decreasing on $[0, \varphi(0)]$.

Lemma 5.1 *Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, \infty]$ such that $\varphi(1) = 0$, and let φ^{-1} be the pseudo-inverse of φ defined by 5.21. Let C be the function from $\mathbb{I}^2$ to $\mathbb{I}$ given by,*

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)). \tag{5.23}$$

Then C satisfies the boundary conditions (5.6) for a copula.

Proof: Refer [31] page 90.

There are several Archemedian copulas that can be constructed by the above procedure. But, the most popular Archemedian copulas are Gumbel, Frank, and Clayton copulas. They are given as,

Definition 5.15 (Gumbel Copula)

$$C_{\theta}^{Gu}(u, v) = \exp\left\{-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right\}, \qquad 1 \le \theta < \infty. \tag{5.24}$$

Definition 5.16 (Clayton Copula)

$$C_{\theta}^{Cl}(u, v) = \max\left\{\left[u^{-\theta} + v^{\theta} - 1\right]^{-\frac{1}{\theta}}, 0\right\}, \qquad 0 < \theta < \infty. \tag{5.25}$$

Definition 5.17 (Frank Copula)

$$C_{\theta}^{Fr}(u, v) = -\frac{1}{\theta} \ln\left(1 - \frac{(e^{-\theta u} - 1)(e^{\theta v} - 1)}{e^{-\theta} - 1}\right), \qquad \theta \in \mathbb{R}. \tag{5.26}$$

Multivariate Archemedian Copulas

The extension of bivariate Archemedian copulas to a d-dimensional case it not straightforward. A necessary condition for an Archemedian copula in d-dimensions to exist is given by Kimberling [34] and in [30]. This condition states that for a d-dimensional Archemedian copula to exist the function φ should be a strict Archemedian copula generator. But, this is not a sufficient condition.

Theorem 5.5 (Kimberling) *Let* $\varphi : [0,1] \mapsto [0,\infty]$ *be a strict Archemedian copula generator, then*

$$C(u_1, u_2, \ldots, u_d) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_d)), \tag{5.27}$$

gives a copula in d-dimensions if and only if the φ^{-1} *is* **completely monotonic.**

A decreasing function $f(t)$ is completely monotonic on an interval $[a, b]$ if it satisfies,

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \ k \in \mathbb{N}, t \in (a, b). \tag{5.28}$$

Generators of Clayton, Frank, and Gumbel copulas are completely monotonic on $[0, \infty]$. For example a d-dimensional Clayton copula is given as,

$$C_\theta^{Cl}(u_1, u_2, \ldots, u_d) = (u_1^{-\theta} + u_2^{-\theta} + \cdots + u_d^{-\theta} - d + 1)^{\frac{1}{\theta}}, \ \theta \geq 0. \tag{5.29}$$

Although Archemedian copulas have numerous application, its applicability is limited to low dimensional data. Since the estimation of high dimensional Archemedian copula is computationally inefficient and very few methods exist to calibrate a general Archemedian copula [35].

5.4 Calibration of Copula functions

There are a few popular methods used to calibrate a copula to real market data. Most of these methods involve direct maximization of log-likelihood function of the copula density with respect to the parameters. Furthermore, the calibration of high dimensional Archemedian copulas and Normal mixture copulas is at the forefront of research. The conditional estimation method for Archemedian copulas was proposed by Marshal and Olkin [35] and Frees and Valdez [36]. Marshal and Olkin [35] also proposed an algorithm

to draw samples from Archemedian copulas. However, here we shall concentrate only on estimation of symmetric elliptical copulas. As these copulas have applications in risk management and VaR estimation.

One method of estimating the parameters of a copula is to directly maximize the likelihood function. Let Θ be the space of parameters with parameters θ. Let $X_d^t = (X_1^t, X_2^t, \ldots, X_d^t)$ represent a sample point of d underlying assets at time t. Then the time series $X = (X_1^t, X_2^t, \ldots, X_d^t)_{t=1}^T$ represents time trajectory of the entire portfolio consisting d assets. Also, let $\mathcal{L}_i(\theta)$ and $\ell_i(\theta)$ be the likelihood and the log-likelihood functions of the underlying copula for the ith sample. Then,

$$\ell(\theta) = \sum_{i=1}^{i=T} \ln c(F_1(x_1^i), \ldots, F_d(x_d^i)) + \sum_{i=1}^{T} \sum_{n=1}^{d} \ln(f_n(x_n^i)). \qquad (5.30)$$

Now, we maximize $\hat{\theta}$ with respect to (5.30) as,

$$\hat{\theta} = \arg\max\{\ell(\theta) | \theta \in \Theta\}.$$

Thus for the Gaussian copula the log-likelihood function takes the form,

$$\ell^{Ga}(\theta) = -\frac{T}{2} \ln |R| - \frac{1}{2} \sum_{i=1}^{T} \zeta_i'(R^{-1} - I)\zeta_i, \qquad (5.31)$$

where $\theta = \{R | R \in \mathbb{R}^{d \times d}\}$ and R is a symmetric positive definite matrix. Thus as it can be seen the estimation of a Gaussian copula is simple and the parameters also have a closed form solution.

Similarly, were can write $\ell(\theta)$ for the Student's t copula as,

$$
\begin{aligned}
\ell^{St}(\theta) &= T \ln\left(\frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})}\right) - dT \ln\left(\frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})}\right) - \frac{T}{2} \ln |R| \\
&\quad - \frac{\nu+d}{2} \sum_{i=1}^{T} \ln(1 + \frac{\zeta_i' R^{-1} \zeta_i}{\nu}) + \frac{\nu+1}{2} \sum_{i=1}^{T} \sum_{n=1}^{d} \ln(1 + \frac{\zeta_{ni}^2}{\nu}),
\end{aligned}
\qquad (5.32)
$$

where $\theta = \{(\nu, R) | \nu \in (2, \infty], R \in \mathbb{R}^{d \times d}\}$. But, as it can be seen, the maximization of log-likelihood function with respect to the parameters is complicated as it involves simultaneous estimation of the parameters of the dependence structure and the margins. Furthermore, this procedure requires a huge amount of data [37]. A more efficient method known as the canonical maximum likelihood method (CML) is proposed by Mashal and Zeevi [38]. We discuss this method in Section (5.4.2).

5.4.1 Inference Functions for Margins method (IFM)

This method was suggested by Joe and Xu [39]. It exploits the structure of (5.30) for efficient estimation of copula parameters. Let us rewrite (5.30) to see the method by which its structure can be exploited,

$$\ell(\theta) = \sum_{i=1}^{i=T} \ln c(F_1(x_1^i; \theta_1), \ldots, F_d(x_d^i; \theta_d); \alpha) + \sum_{i=1}^{T} \sum_{n=1}^{N} \ln(f_n(x_n^i)). \tag{5.33}$$

This method particularly exploits the separation between univariate marginals $\theta = \{\theta_1, \theta_2, \ldots, \theta_d\}$ and the vector of copula parameters α appearing in (5.33). The estimation is the beautifully carried out in two discrete steps as follows,

1. Estimate the univariate marginals (θ_n's) via the EML method. Thus,

$$\hat{\theta}_n = \arg\max_{\theta_i} \sum_{i=1}^{T} \ln f_n(x_n^i; \theta_n). \tag{5.34}$$

2. In the second step estimate the copula parameter α. This step uses the estimates of θ_n's obtained in Step (1) as follows,

$$\hat{\alpha} = \arg\max_{\alpha} \sum_{i=1}^{T} \log c(F_1(x_1^i; \hat{\theta}_1), \ldots, F_d(x_d^i; \hat{\theta}_d); \alpha). \tag{5.35}$$

Thus, the above two step gives us all the parameters of a copula for real data.

5.4.2 Canonical Maximum Likelihood Method (CML)

Both the methods described above make assumptions on the distribution of the marginals while they calibrate copulas to real data. However, the CML method does no make any distributional assumption. It uses a transformation known as the *empirical marginal transformation* to transform data to an estimate of the empirical distribution. This estimate of the empirical distribution approximates the unknown marginal distribution $F_n(\cdot)$ as,

$$\hat{F}_n(\cdot) \triangleq \frac{1}{T} \sum_{t=1}^{T} 1_{\{X_{nt} \leq \cdot\}}, \tag{5.36}$$

where $1_{\{X_{nt} \leq \cdot\}}$ is an indicator function. Thus the empirical marginal transformation transforms the data to uniform variates via (5.36). Using this transformation the CML method can be summarized in the following two steps:

1. Transform the real data set $X = (X_{1t}, X_{2t}, \ldots, X_{dt})_{t=1}^{T}$ into uniform variates using (5.36). Thus, $(\hat{u}_1^t, \hat{u}_2^t, \ldots, \hat{u}_d^t) = [\hat{F}_1(X_{1t}), \hat{F}_2(X_{2t}), \ldots, \hat{F}_d(X_{Nt})])$;

2. Estimate the copula parameters vector α using the following relation,

$$\hat{\alpha} = \arg\max_{\alpha} \sum_{t=1}^{T} \ln c(\hat{u}_1^t, \hat{u}_2^t, \ldots, \hat{u}_d^t; \alpha). \tag{5.37}$$

The CML method is the most efficient method in terms of computations amongst all the methods described above. Thus CML method is the most popular estimation technique. In this method numerical maximization is only induced in Step (2). This is the primary reason for the CML method for being computationally efficient [38].

Mashal *et al.* [38] show the effective application of the CML method for calibrating a Student's t copula. They propose an estimation procedure that uses the rank correlation estimator, particularly the Kendall's τ estimator. Thus let us define the Kendall's τ. Kendall's τ for a random vector (X, Y) of length T is defined as [30],

$$\tau(X, Y) = P\{(X - \overline{X}) \cdot (Y - \overline{Y}) > 0\} - P\{(X - \overline{X}) \cdot (Y - \overline{Y}) < 0\}, \tag{5.38}$$

where $(\overline{X}, \overline{Y})$ is an identical copy of the original vector (X, Y). The sample estimator of τ, $\hat{\tau}$ is given by [40]. Let c denote the number of pairs such that $(x_i - x_j) \cdot (y_i - y_j) > 0$ and d denote the number of pairs such that $(x_i - xj) \cdot (y_i - y_j) < 0$ [1]. Then,

$$\hat{\tau} = \frac{c - d}{c + d}. \tag{5.39}$$

Using the estimate of τ in (5.39) we can obtain a robust and efficient estimator of the matrix R. This is due to theorem by Lindskog *et al.* [41].

Theorem 5.6 *Let $X \sim \mathfrak{E}_d(\mu, \Sigma, \varphi)$ where for $i, j \in \{1, 2, \ldots, d\}$, X_i and X_j are continuous. Then,*

$$R_{ij} = \sin(\frac{\pi}{2}\tau(X_i, X_j)), \tag{5.40}$$

where $\mathfrak{E}_d(\mu, \Sigma, \varphi)$ denotes the d − dimensional elliptical distribution with parameters (μ, Σ, φ), and $\tau(X_i, X_j)$ and R_{ij} indicate, respectively, the Kendall's τ and the Pearson's linear correlation coefficient for the random variables X_i and X_j.

Proof: Refer [41]

Using Theorem (5.6) the CML method is summarized in Algorithm ([?]):

---

[1]The adjustment in case of ties is given in [40]

Algorithm 5.1 (calibrating a Student's *t* copula using the CML method)

1. *Transform the data X_n to pseudo-samples U_n using the empirical marginal transformation;*

2. *Estimate the correlation matrix R_{ij} using Theorem (5.6);*

3. *Perform the unconstrained maximization for $\hat{\nu}$ as,*

$$\hat{\nu} = \arg \max_{\nu \in (2,\infty]} \sum_{i=1}^{T} \log c(\hat{u}_1^t, \hat{u}_2^t, \ldots, \hat{u}_d^t; R, \nu). \tag{5.41}$$

The above method has computational advantages over other methods [38]. The estimator for R_{ij} is $O(n^2)$ for each coefficient. This is better that the iterative procedure for estimation. And, in our experience as the maximization is carried out with respect to only one parameter (ν) it is significantly faster and stable than maximization with respect to several parameters.

5.5 Sampling from Copulas

One of the primary applications of copulas is in generating samples from a specified joint distribution. Such samples can then be used for Monte Carlo studies [42, 43]. Copulas offer an elegant and efficient method from sampling from multivariate distributions with a given dependence structure. These sampling methods have numerous applications in CDO pricing, n^{th} to default swap pricing, and pricing many other credit derivatives [33]. Here we first describe a generic algorithm for generating random variates from a joint distribution $H(x, y)$. Later, in this section we also discuss algorithms for sampling from well known elliptical copulas.

There are a variety of ways for generating observations (x, y) of a pair of random variables (X, Y) with a joint distribution function $H(x, y)$. Here we describe a copula based method. By the virtue of Sklar's theorem (5.3), we only need to generate a pair (u, v) of observations of uniform $(0, 1)$ random variables (U, V) whose joint distribution function is C, the copula of X and Y, and then transform these uniform variates to (x, y) by the method described in Algorithm (5.2).

Algorithm 5.2 (Generate random variates from $C(F(x), F(y))$)

1. *Generate two independent uniform* $(0,1)$ *variates* u *and* t;

2. *Set* $v = c_u^{(-1)}(t)$, *where* $c_u^{(-1)}$ *denotes the quasi-inverse of* $c_u(v) = \frac{\partial}{\partial u} C(u,v)$;

3. *You have* (u,v), *use* $x = F^{-1}(u)$ *and* $y = F^{-1}(v)$ *to obtain* (x,y) *(see 5.9);*

4. *Output* (x,y), *a desired sample from joint distribution* $H(x,y)$.

Thus, by Algorithm (5.2) we can generate random variates having joint distribution $H(x,y)$. Now we shall describe algorithms for sampling from various well known elliptical copulas.

Algorithm 5.3 (Sampling from Multivariate Gaussian Copula)

1. *Decompose the matrix* $R = A \cdot A^T$ *using the Cholesky decomposition;*

2. *Draw an* N*-dimensional uncorrelated vector* $z = (z_1, z_2, \ldots, z_N)'$ *of standard normal variates;*

3. *Set* $x = z^T A$, *and transform* x *to an* N*-dimensional uniform variate on* $[0,1]$ *by computing* $u = \Phi(x)$;

4. *Then,* $u \sim C_R^{Ga}$.

Algorithm 5.4 (Sampling from Student's t Copula)

1. *Decompose the matrix* $R = A \cdot A^T$ *using the Cholesky decomposition;*

2. *Draw an* N*-dimensional uncorrelated vector* $z = (z_1, z_2, \ldots, z_N)'$ *of standard normal variates;*

3. *Draw an independent random variable* $s \sim \chi_\nu^2$;

4. *Set* $x = (\sqrt{\frac{\nu}{s}}) z^T A$, *and transform* x *to an* N*-dimensional uniform variate on* $[0,1]$ *by computing* $u = t_\nu(x)$;

5. *Then,* $u \sim C_{R,\nu}^{St}$.

We used the above algorithms to draw correlated samples from various copulas. Figure (5.1) and Figure (5.2) shows 4000 samples drawn from the Gaussian and the Student's t copula with various parameters. It is interesting to note that the samples drawn from the Student's t copula exhibit *heavy-tails* as compared to the samples drawn from a Gaussian copula. Furthermore, as the value of dependence parameter ρ increases for the Student's t copula the tails become heavier.
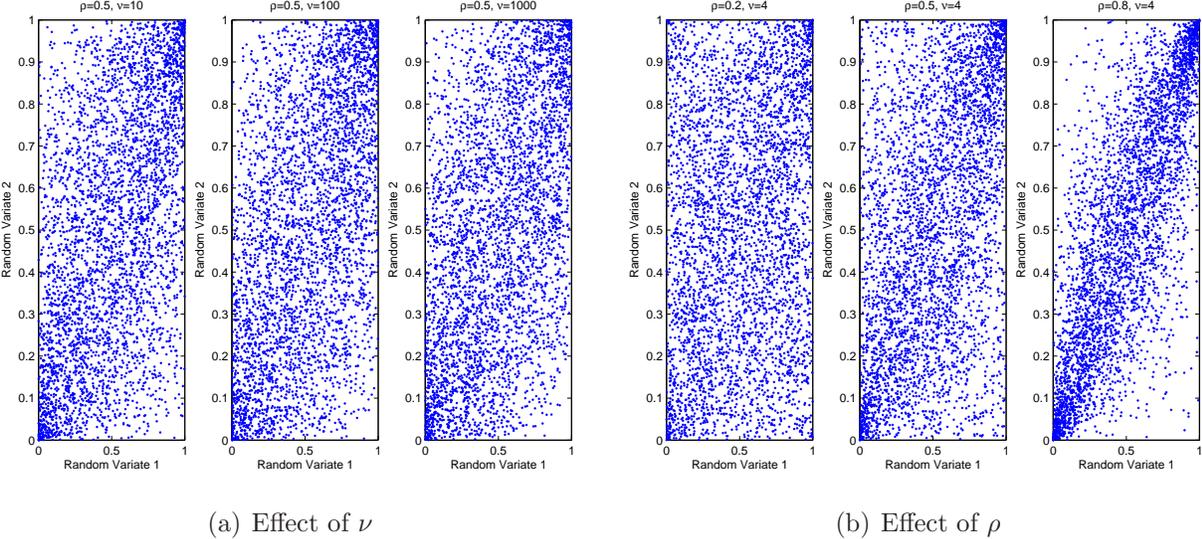

(a) Effect of ν (b) Effect of ρ

Figure 5.1: Samples from a 4000 run Student's t copula highlighting effects of various parameters.
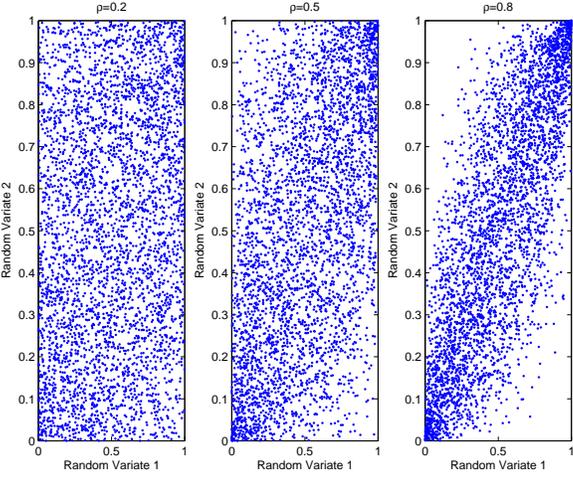

Figure 5.2: Samples from a 4000 run Gaussian copula highlighting effects of parameter ρ.

5.6   Application to Risk Management

One of the major application of copulas is in modeling extreme co-movements in portfolios of assets. These extreme movements quantify risk in the underlying portfolio. In Chapter 4 we have already defined *Value-at-Risk (VaR)*, but the definition given there was for a univariate case (single asset). However, *VaR* of a portfolio of assets can also be defined in a similar way, by first obtaining a linear combination of all the names in a portfolio and then applying the original definition provided in Chapter 4. Thus, portfolio *VaR* involves estimation of joint distribution of assets in a portfolio. Let the value of the portfolio under consideration at time t be,

$$P_t = a_1 X_{1t} + a_2 X_{2t} + \cdots + a_d X_{dt}, \qquad \text{where } \sum_{i=1}^d a_i = 1 \text{ and } a_i > 0, \qquad (5.42)$$

where $X = (X_{1t}, X_{2t}, \ldots, X_{dt})_{t=1}^T$ represent the price time series of assets in the portfolio, and $a_1, \ldots, a_d$ are weights by which the assets are combined. Now, the method for computing *VaR* of a portfolio is given in Algorithm (5.5).

Algorithm 5.5 (portfolio *VaR*)

1. *Calculate the return series of individual assets by (2.1). Let this return series be represented as, $(R_{1t}, R_{2t}, \ldots, R_{dt})_{t=1}^T$;*

2. *Calibrate a copula (Gaussian, Student's t,etc) to $(R_{1t}, R_{2t}, \ldots, R_{dt})_{t=1}^T$ by any of the methods discussed in Section (5.4);*

3. *Draw sufficient number of samples from the copula calibrated in Step (2). These samples represent returns following a joint distribution as modeled by the copula;*

4. *Combine these samples to from return on a portfolio as,*

$$R_t^p = a_1 R_{1t} + a_2 R_{2t} + \cdots + a_d R_{dt}; \qquad (5.43)$$

5. *Estimate VaR of R_t^p as discussed in Chapter 4.*

5.7   Numerical Results

We use Algorithm (5.5) to estimate *VaR* in various portfolios. The portfolios used are shown in Table (5.1). As it can be observed from Table (5.3) these portfolios comprise of

various number and types of assets. We calibrate Gaussian and Student's t copula to these four portfolios and then estimate *VaR*. The degrees of freedom (ν) versus log-likelihood plots for a Student's t copula are shown in Figure (5.3). The value of ν is chosen to be that value of ν where the log-likelihood is maximum.

The *VaR* estimates obtained from various copulas are shown in Table (5.2). Clearly, *VaR* estimates obtained from Student's t copula assumption are better than the ones obtained from the Gaussian copula assumption. This can be attributed to Student's t copula being heavier in tails than the Gaussian copula. But none of the copulas produce estimates having good approximation to the empirical *VaR*. Thus although the Student's t copula provides better estimates than the Gaussian copula it is not sufficient to model the *heavy-tails* present in our portfolios. Thus in the next chapter we shall use multivariate Generalized Hyperbolic distributions to supersede the estimates obtained here.

Table 5.1: Portfolios used for *VaR* estimation

Symbol	NSEIND	PORTF-5	PORTF-10	PORTF-15
Samples	1482	904	904	904
No. of assets	4	5	10	15
Names	NIFTY, NIFTY JUNIOR, CNX500, CNXMIDCAP	BEML, BHEL, EIDPARRY, GRASIM, HDFC	BEML, BHEL, EIDPARRY, GRASIM, HDFC, INFOSYSTCH, M&M, SAIL, SEIMENS, SUNPHARMA	BEML, BHEL, EIDPARRY, GRASIM, HDFC, INFOSYSTCH, M&M, SAIL, SEIMENS, SUNPHARMA TELCO, TV18, TAJGVK, PNB, BHARTI

Table 5.2: *VaR* estimates obtained from various copulas

Portfolio	Empirical	Gaussian	Student's t
NSEIND	6.0470	2.1483	4.3013
PORTF-5	5.3432	1.2407	1.6625
PORTF-10	5.6905	0.9946	1.4367
PORTF-15	4.4035	1.0162	1.4346

Table 5.3: Assets used for *VaR* estimation

Equity Asset	Symbol
Bharat Earth Movers Ltd.	BEML
Bharat Heavy Electrical Ltd.	BHEL
EID Parry (India) Ltd.	EIDPARRY
Grasim Industries	GRASIM
HDFC Ltd.	HDFC
Infosys Technologies Ltd.	INFOSYSTCH
Mahindra and Mahindra	M&M
Steel Authority of India Ltd.	SAIL
Siemens Ltd.	SEIMENS
Sun Pharmaceuticals Industries Ltd.	SUNPHARMA
Tata Motors Ltd.	TELCO
Television Eighteen India Ltd.	TV18
TajGVK Hotels and Resorts Ltd.	TAJGVK
Punjab National Bank	PNB
Bharti Tele Ventures Ltd.	BHARTI
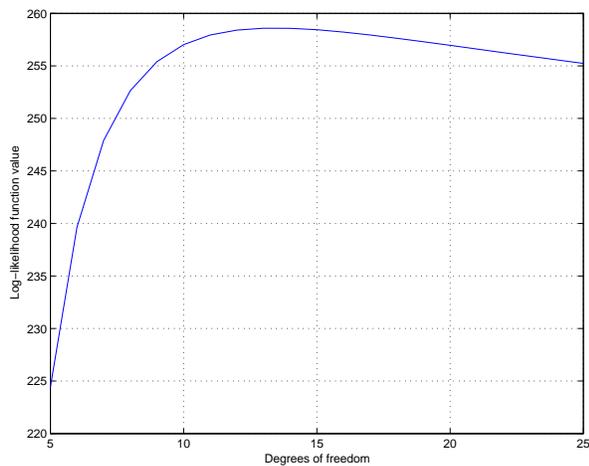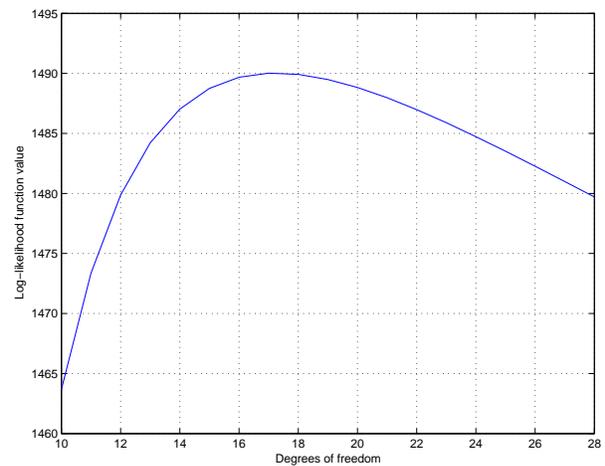
(a) PORTF-5 ($\nu = 14$) (b) PORTF-15 ($\nu = 17$)

Figure 5.3: Degrees of Freedom (DoF) ν vs. Log-likelihood for Student's t copula.

Chapter 6

Multivariate Mixtures

Recently, there has been growing interest in non-elliptical (hyperbolic) distributions as these distributions are heavier in tails and have the ability to capture extreme (co)-movements accurately [6]. In Chapter 4 we saw how univariate Generalized Hyperbolic distributions produce significantly better risk estimates than their elliptical counterparts. In this chapter we introduce multivariate Generalized Hyperbolic distributions and describe an Expectation-Maximization (EM) type algorithm to estimate their parameters. Lastly, we calibrate these distributions to real market portfolios given in Table (5.1) and compare *VaR* estimates with those obtained from copulas. We conclude that *VaR* estimates obtained form the multivariate Generalized Hyperbolic assumption are significantly superior to the estimates obtained from copulas.

6.1   Normal Variance Mixtures

Definition 6.1 *The random vector X is said to have a (multivariate) normal variance mixture distribution if*

$$X \triangleq \mu + \sqrt{W} AZ, \tag{6.1}$$

where $Z \sim N_k(0, I_k)$, $W \geq 0$ is a non-negative scalar random variable which is independent of Z, and $A \in \mathbb{R}^{d \times k}$ and $\mu \in \mathbb{R}^d$ are respectively a matrix and vector of constants.

These distributions are known as variance mixtures because $(X|W) = w \sim N_d(\mu, w\Sigma)$, where $\Sigma = AA'$. Such distributions can be thought of as being constructed from many multivariate normal distributions.

Definition 6.2 (Generalized Inverse Gaussian) *The random variable X has a generalized inverse Gaussian (GIG) distribution $N^-(\lambda, \chi, \phi)$, if its density is*

$$f(x) = \frac{\chi^{-\lambda}(\sqrt{\chi\psi})^\lambda}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} \exp(-\frac{1}{2}(\chi x^{-1} + \psi x)), \qquad x > 0, \tag{6.2}$$

where K_λ denotes a modified Bessel function of the third kind with index λ and the parameters satisfy $\chi > 0$, $\psi \geq 0$, if $\lambda < 0$; $\chi > 0, \psi > 0$ if $\lambda = 0$; and $\chi \geq 0$, $\psi > 0$ if $\lambda > 0$.

6.2 Normal Mean-Variance Mixtures

The multivariate distributions studied until now have elliptical symmetry and thus provide a oversimplified model for risk. But as it is well known that negative returns (losses) have heavier tails than positive returns (gains) it is necessary that we introduce some kind of asymmetry in our models. The multivariate normal mean-variance mixtures attempt to add some asymmetry to the class of normal mixtures by mixing normal distributions with different means as well as different variances.

Definition 6.3 *The random variable X has a (multivariate) normal mean-variance mixture distribution if*

$$X = m(W) + \sqrt{W}AZ, \tag{6.3}$$

where

- $Z \sim N_k(0, I_k)$;

- $W \geq 0$ *is a non-negative, scalar-valued random variable which is independent of Z;*

- $A \in \mathbb{R}^{d \times k}$ *is a matrix; and*

- $m : [0, \infty) \mapsto \mathbb{R}^d$ *is a measurable function.*

One possible representation of $m(W)$ is,

$$m(W) = \mu + W\gamma, \tag{6.4}$$

where μ and γ are parameters of $\mathbb{R}^d$.

Here we have,

$$(X|W) = w \sim N_d(m(w), w\Sigma), \tag{6.5}$$

where $\Sigma = AA'$. In general these distributions are not elliptical since $E(X|W) = \mu + W\gamma$ and $cov(X|W) = W\Sigma$, it follows in this case by simple calculations that,

$$E(X) \;=\; E(E(X|W)) = \mu + E(W)\gamma, \tag{6.6}$$

$$cov(X) \;=\; E(cov(X|W)) + cov(E(X|W)), \tag{6.7}$$

$$\;=\; E(W)\Sigma + var(W)\gamma\gamma',$$

where the mixing variable W has finite variance.

6.3   Generalized Hyperbolic Distribution

A Generalized Hyperbolic distribution is a normal mean-variance mixture with a GIG (Generalized Inverse Gaussian) with parameters $W \sim N^{-}(\lambda, \chi, \psi)$ as a mixing distribution.

Definition 6.4 *The joint density of d-dimensional Generalized Hyperbolic distribution in a non-singular case (Σ has rank d) and with mixing variable $W \sim N^{-}(\lambda, \chi, \psi)$ is given by,*

$$f(x) = c \frac{K_{\lambda-\frac{d}{2}}(\sqrt{(\chi + (x-\mu)'\Sigma^{-1}(x-\mu))(\psi + \gamma'\Sigma^{-1}\gamma)}) \exp((x-\mu)'\Sigma^{-1}\gamma)}{(\sqrt{(\chi + (x-\mu)'\Sigma^{-1}(x-\mu))(\psi + \gamma'\Sigma^{-1}\gamma)})^{\frac{d}{2}-\lambda}}, \tag{6.8}$$

where the normalizing constant c is,

$$c = \frac{(\sqrt{\chi\psi})^{-\lambda}\psi^{\lambda}(\psi + \gamma'\Sigma^{-1}\gamma)^{\frac{d}{2}-\lambda}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{1/2}K_{\lambda}(\sqrt{\chi\psi})}, \tag{6.9}$$

and $|\cdot|$ denotes determinant.

If $\gamma = 0$ then the above distribution reduces to the symmetric Generalized Hyperbolic distribution.

Proposition 6.1 *If $X \sim GH_d(\lambda, \chi, \psi, \mu, \Sigma, \gamma)$ and $Y = BX + b$ where $B \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$, then $Y \sim GH_k(\lambda, \chi, \psi, B\mu + b, B\Sigma B', B\gamma)$.*

   Proof: Refer [7]

Corollary 6.1 *If $B = \omega^T = (\omega_1, \omega_2, \ldots, \omega_d)$ and $b = 0$, then the portfolio $y = \omega^T X$ is a one dimensional Generalized Hyperbolic distribution of the form,*

$$y \sim GH_1(\lambda, \chi, \psi, \omega^T\mu, \omega^T\Sigma\omega, \omega^T\gamma). \tag{6.10}$$

The above corollary also shows that once the multivariate Generalized Hyperbolic distribution is calibrated its marginals are obtained automatically as, $X_i \sim GH_1(\lambda, \chi, \psi, \mu_i, \Sigma_{ii}, \gamma_i)$.

Thus it can be observed that just like Gaussian and Student's t distribution the Generalized Hyperbolic distribution is also closed under linear transformations. This corollary has nice implications in risk management. It implies that the methods used to estimate risk under the Gaussian or the Student's t assumption in the case of portfolio can also be used in the case of Generalized Hyperbolic distributions. Although, the Generalized Hyperbolic distributions are not elliptical like Gaussian and Student's t distribution. The non-ellipticity is useful to capture more characteristics about financial data such as asymmetry and *heavy-tailedness*.

6.4 Parameterizations

There are various different parameterizations in existence for the Generalized Hyperbolic distributions. All these parameterizations have unique applications and highlight different features of this rich family of distributions.

The most popular parameterization was introduced by Blasild [44]. Here the dispersion matrix Σ is renamed to Δ and a constraint is imposed that $|\Delta| = 1$. The remaining parameters are replaced as below,

$$
\begin{aligned}
\beta &= \Delta^{-1}\gamma, \\
\delta &= \sqrt{\chi}, \\
\alpha &= \sqrt{\psi + \gamma'\Delta^{-1}\gamma}.
\end{aligned}
\tag{6.11}
$$

This parameterization must satisfy the following constraints,

$$
\begin{aligned}
\delta &\geq 0, \alpha^2 > \beta'\Delta\beta, \ \text{if } \lambda > 0, \\
\delta &> 0, \alpha^2 > \beta'\Delta\beta, \ \text{if } \lambda = 0, \\
\delta &> 0, \alpha^2 \geq \beta'\Delta\beta, \ \text{if } \lambda < 0.
\end{aligned}
\tag{6.12}
$$

It is desirable to know the relations which allow us to move between the $\chi - \psi - \Sigma - \gamma$ parametrization and the $\alpha - \delta - \Delta - \beta$ parameterization. It is valuable to note that λ

and μ are common to both the parameterizations. The remaining relations are given as follow,

$$
\begin{aligned}
\Delta &= |\Sigma|^{-1/d}\Sigma, \\
\beta &= \Sigma^{-1}\gamma, \\
\delta &= \sqrt{\chi|\Sigma|^{1/d}}, \\
\alpha &= \sqrt{|\Sigma|^{-1/d}(\psi + \gamma'\Sigma^{-1}\gamma)}.
\end{aligned}
\tag{6.13}
$$

If the $\alpha - \delta - \Delta - \beta$ parameterization is used then we can get the $\chi - \psi - \Sigma - \gamma$ by,

$$
\begin{aligned}
\Sigma &= \Delta, \\
\gamma &= \Delta\beta, \\
\chi &= \delta^2, \\
\psi &= \alpha^2 - \beta'\Delta\beta.
\end{aligned}
\tag{6.14}
$$

6.5 Special cases of the Generalized Hyperbolic distribution

The Generalized Hyperbolic family of distributions has many special cases. The Gaussian, Student's t, Normal Inverse Gaussian, skew Student's t, Hyperbolic, Variance-Gamma distributions are all special cases of the Generalized Hyperbolic distribution. It is evident that this family of distributions is rich and diverse in nature. Some of the special cases are described below [7],

- For $\lambda = \frac{1}{2}(d+1)$ we get the Hyperbolic distribution. The univariate margins of this distribution also have $\lambda = \frac{1}{2}(d+1)$;

- For $\lambda = 1$ we get a distribution whose univariate margins are one-dimensional hyperbolic distributions;

- For $\lambda = -\frac{1}{2}$ we get another flexible and popular family of distribution known as the Normal Inverse Gaussian distribution (NIG). It has slightly heavier tails than the Hyperbolic distribution;

- For $\lambda > 0$ and $\chi = 0$ we get a family of distributions known as the Generalized Laplace distributions or Variance-Gamma distributions;

- For $\lambda = -\frac{1}{2}$, $\chi = \nu$, and $\psi = 0$ we get a limiting family for distributions called the asymmetric or skewed t distribution. These are not a very well studied family of distributions [45]. Evaluating the limit of (6.8) as $\psi \to 0$ yields,

$$f(x) = c \frac{K_{\frac{\nu+d}{2}}(\sqrt{(\nu + (x-\mu)'\Sigma^{-1}(x-\mu))(\gamma'\Sigma^{-1}\gamma)})\exp((x-\mu)'\Sigma^{-1}\gamma)}{(\sqrt{(\nu + (x-\mu)'\Sigma^{-1}(x-\mu))(\gamma'\Sigma^{-1}\gamma)})^{-\frac{\nu+d}{2}}(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu})^{\frac{\nu+d}{2}}}, \quad (6.15)$$

where the normalizing constant is,

$$c = \frac{2^{1-(\nu+d)/2}}{\Gamma(\frac{1}{2}\nu)(\pi\nu)^{d/2}|\Sigma|^{\frac{1}{2}}}. \quad (6.16)$$

As $\gamma \to 0$ the above density reduces to the standard multivariate Student's t density.

6.6 Calibration of Multivariate Generalized Hyperbolic distribution (MGHD)

In Section (4.2) we saw a maximum likelihood type method for calibration of univariate Generalized Hyperbolic distributions. There direct maximization of the log-likelihood function was possible because all the parameters involved were scalars. This problem becomes challenging in the multivariate case as here the log-likelihood function involves vectors and matrices. One of the first attempts to calibrate a multivariate Generalized Hyperbolic distribution was made by Prause [6]. He successfully calibrated a 3-dimensional MGHD. As it can be seen the problem of calibration of multivariate Generalized Hyperbolic distribution is recent. Later, Prostassov [46] gave an algorithm which can theoretically be used to calibrate a MGHD with any dimension but in practice suffered from instability while calibrating high dimensional MGHD. Prostassov [46] used his proposed algorithm to calibrate a 5-dimensional NIG distribution. The most recent algorithm in this area is given by McNeil *et al.* [7], which they claim to be stable than the algorithm proposed in [46]. We use this algorithm to calibrate a 15-dimensional MGHD. Lastly, we compare the *VaR* estimates obtained from MGHDs and copulas.

Initially we shall take a look at the ML estimation problem of Generalized Hyperbolic distributions. Let $X_1, X_2, \ldots, X_n$ be the point to which we wish to fit a Generalized

Hyperbolic distribution. The parameter vector θ comprise of $\theta = (\lambda, \chi, \psi, \mu, \Sigma, \gamma)'$. Thus the problem is,

$$\ell(X_1, X_2, \ldots, X_n; \theta) = \arg\max_{\theta} \sum_{i=1}^{n} \log f_X(X_i; \theta), \tag{6.17}$$

where $f_X(X_i; \theta)$ denotes the Generalized Hyperbolic density. As it can be seen the maximization of the above equation is not straight forward since it involves a matrix Σ and several other parameters. This maximization also suffers from a plethora of numerical problems. The problem of estimation of high-dimensional (10-12 dimensional) densities is even more difficult and computationally challenging. Apart from a few algorithms proposed recently, there are no satisfactory estimation algorithms for the Generalized Hyperbolic distributions. Estimation of MGHD is still an active area of research.

Prostassov [46] observed that if we could observe the latent mixing variables then the representation in (6.17) would be much simpler. Thus we can now model the joint density as,

$$f_{X,W}(x, w; \theta) = f_{X|W}(x|w; \mu, \Sigma, \gamma) h_W(w; \lambda, \chi, \psi). \tag{6.18}$$

And the log-likelihood is given as,

$$\ell(X_1, \ldots, X_n, W_1, \ldots, W_n; \theta) = \sum_{i=1}^{n} \log f_{X|W}(X_i|W_i; \mu, \Sigma, \gamma) + \sum_{i=1}^{n} h_W(W_i; \lambda, \chi, \psi). \tag{6.19}$$

Now, the two terms in (6.19) can be maximized separately. There is also an EM type algorithm proposed to estimate the parameters and to overcome latency of W_i's [7].

The EM algorithm proceeds as follows, suppose at the beginning of step k we have parameter estimates $\theta^{[k]}$. The algorithm then proceeds as follows,

E-step: Given the data $X_1, \ldots, X_n$ and the parameters $\theta^{[k]}$ we compute the conditional expectation of the likelihood function given in (6.19) as,

$$Q(\theta, \theta^{[k]}) = E(\ell(X_1, \ldots, X_n, W_1, \ldots, W_n)|X_1, \ldots, X_n; \theta^{[k]})); \tag{6.20}$$

M-step: Maximize the above equation with respect to θ to obtain $\theta^{[k+1]}$.

Executing the above steps iteratively produces improved parameter estimates at every step. However, the EM algorithm does not guarantee global convergence.

The M-step involves maximization of two terms which come from (6.19). These are indicated as $\ell_1(\mu, \Sigma, \gamma; \theta^{[k]})$ and $\ell_2(\lambda, \chi, \psi; \theta^{[k]})$. This results in a so-called MCECM algorithm (multi-cycle, expectation, conditional maximization), which is presented below.

Algorithm 6.1 (EM estimation of multivariate Generalized Hyperbolic distribution)

(1) *Set the starting values for* μ, γ *and* Σ, *respectively to sample mean, zero vector and sample covariance matrix S. Also, set k=1;*

(2) *Calculate weights* $\delta_i^{[k]}$, $\eta_i^{[k]}$, *and* $\xi_i^{[k]}$ *as,*

$$\delta_i^{[k]} = \left(\frac{\rho_i^{[k]} + \chi^{[k]}}{\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]}} \right)^{-\frac{1}{2}} \frac{K_{\lambda-(d/2)-1}\sqrt{(\rho_i^{[k]} + \chi^{[k]})(\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]})}}{K_{\lambda-(d/2)}\sqrt{(\rho_i^{[k]} + \chi^{[k]})(\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]})}},$$

$$(6.21)$$

$$\eta_i^{[k]} = \left(\frac{\rho_i^{[k]} + \chi^{[k]}}{\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]}} \right)^{\frac{1}{2}} \frac{K_{\lambda-(d/2)+1}\sqrt{(\rho_i^{[k]} + \chi^{[k]})(\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]})}}{K_{\lambda-(d/2)}\sqrt{(\rho_i^{[k]} + \chi^{[k]})(\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]})}},$$

$$(6.22)$$

$$\xi_i^{[k]} = \frac{1}{2}\log\left(\frac{\rho_i^{[k]} + \chi^{[k]}}{\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]}} \right) + \left.\frac{\partial K_{\lambda-(d/2)+\alpha}\sqrt{(\rho_i^{[k]} + \chi^{[k]})(\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]})}}{\partial\alpha}\right|_{\alpha=0}$$

$$\times \frac{1}{K_{\lambda-(d/2)}\left(\sqrt{(\rho_i^{[k]} + \chi^{[k]})(\psi^{[k]} + \gamma'^{[k]}\Sigma^{-1[k]}\gamma^{[k]})} \right)}.$$

$$(6.23)$$

There are analytical expressions provided to compute the derivative of a Bessel function [47]. But numerical procedures tend to be easy and can be used to compute this derivative [47]. In general here we do not need to compute $\xi_i^{[k]}$ *since* λ *in our simulations is a constant.*

Now compute the average of the above parameters as,

$$\overline{\delta}^{[k]} = \frac{1}{n}\sum_{i=1}^{n}\delta_i^{[k]}, \qquad \overline{\eta}^{[k]} = \frac{1}{n}\sum_{i=1}^{n}\eta_i^{[k]}, \qquad \overline{\xi}^{[k]} = \frac{1}{n}\sum_{i=1}^{n}\xi_i^{[k]}; \qquad (6.24)$$

(3) *Set* $\gamma^{[k+1]} = 0$ *for a symmetric model. Otherwise,*

$$\gamma^{[k+1]} = \frac{1}{n}\frac{\sum_{i=1}^{n}\delta_i^{[k]}(\overline{X} - X_i)}{\overline{\delta}^{[k]}\overline{\eta}^{[k]} - 1}; \qquad (6.25)$$

(4) *Update estimates of dispersion matrix and mean vector as follows,*

$$\mu^{[k+1]} \;=\; \frac{1}{n}\frac{\sum_{i=1}^{n}\delta_i^{[k]}X_i - \gamma^{[k+1]}}{\overline{\delta}^{[k]}}, \tag{6.26}$$

$$\Psi \;=\; \frac{1}{n}\sum_{i=1}^{n}\delta_i^{[k]}(X_i-\mu^{[k+1]})(X_i-\mu^{[k+1]})' - \overline{\eta}^{[k]}\gamma^{[k+1]}\gamma'^{[k+1]}, \tag{6.27}$$

$$\Sigma^{[k+1]} \;=\; \frac{|S|^{1/d}\Psi}{|\Psi|^{1/d}}; \tag{6.28}$$

(5) *Now, set*

$$\theta^{[k,2]} = (\lambda^{[k]}, \chi^{[k]}, \psi^{[k]}, \mu^{[k+1]}, \Sigma^{[k+1]}, \gamma^{[k+1]})'. \tag{6.29}$$

Also, recompute weights $\delta_i^{[k,2]}, \eta_i^{[k,2]},$ and $\xi_i^{[k,2]}$ by using equations in Step (2).

(6) *The last step is to maximize $\ell_2(\lambda, \chi, \psi; \theta^{[k,2]})$ with respect $\lambda, \chi,$ and ψ where,*

$$\ell_2(\lambda, \chi, \psi; \theta^{[\cdot]}) \;=\; (\lambda-1)\sum_{i=1}^{n}\xi_i^{[\cdot]} - \frac{1}{2}\chi\sum_{i=1}^{n}\delta_i^{[\cdot]} - \frac{1}{2}\psi\sum_{i=1}^{n}\eta_i^{[\cdot]} \tag{6.30}$$

$$- \frac{1}{2}n\lambda\ln(\chi) + \frac{1}{2}n\lambda\ln(\psi) - n\ln(2K_\lambda(\sqrt{\chi\psi}));$$

(7) *Increment iteration counter $k \to k+1$ and go to Step (2).*

Obviously, the above algorithm can be modified to calibrate all special cases of the Generalized Hyperbolic subclass of distributions mentioned in Section (6.5). We use Algorithm (6.1) to calibrate a bivariate Normal Inverse Gaussian (NIG) distribution. The density function obtained from this calibration is compared with Gaussian density and is shown in Figure (6.1) along with the contour plots in Figure (6.2). Similar to the univariate case, it is worth noting that the NIG density has heavier tails than the Gaussian density and is more leptokurtic.

6.7   Numerical Results

We follow the steps given in Algorithm (5.5) but instead of calibrating a copula as in Step (2), we calibrate a multivariate Generalized Hyperbolic distribution with constant λ using Algorithm (6.1). Thus we calibrate the multivariate Normal Inverse Gaussian ($\lambda = -\frac{1}{2}$) and multivariate Hyperbolic distribution ($\lambda = 1$) to real market data. The results obtained from this procedure for portfolios given in Table (5.1) are shown in Table (6.1). Clearly, the *VaR* estimates obtained from multivariate Generalized Hyperbolic (MGHD) assumption
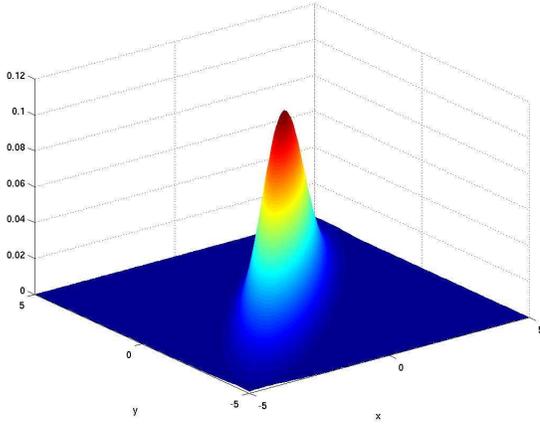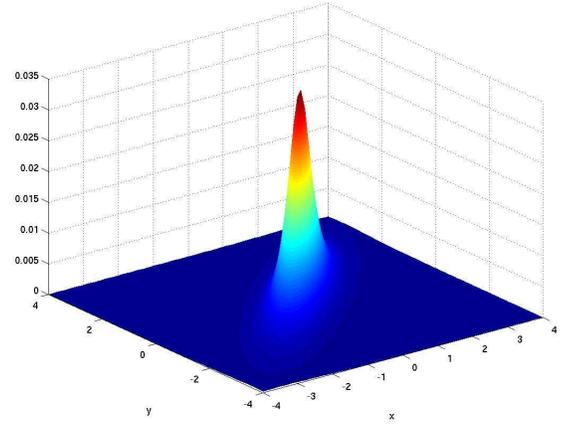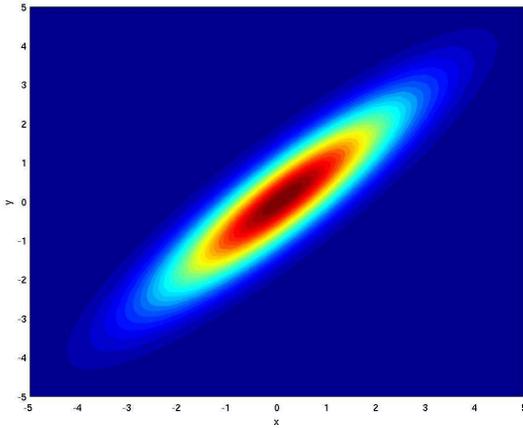
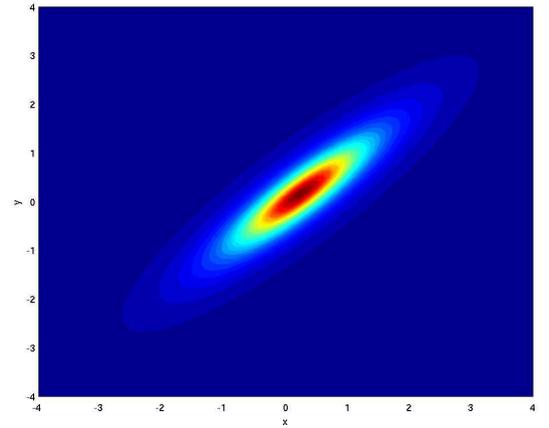(a) Bivariate Gaussian density function   (b) Bivariate Normal Inverse Gaussian density function

Figure 6.1: Comparison of density functions

(a) Bivariate Gaussian density function   (b) Bivariate Normal Inverse Gaussian density function

Figure 6.2: Comparison of contour plots

are significantly better from any of the prior assumptions (refer Table 5.2). Amongst themselves multivariate Normal Inverse Gaussian assumption gives the heaviest tails as compared to all other distributions. But, although accurate the algorithm given above is computationally inefficient and suffers from stability issues for higher dimensions. Further research will be necessary for making this algorithm efficient and even more accurate.

Table 6.1: *VaR* estimates obtained from multivariate NIG and Hyperbolic (HYP) distributions

Portfolio	Empirical	HYP	NIG
NSEIND	6.0470	4.3389	4.6671
PORTF-5	5.3432	4.4517	4.5910
PORTF-10	5.6905	4.0651	4.5116
PORTF-15	4.4035	3.9525	4.4619

Chapter 7

Summary and Conclusion

We have shown that the NSE indices are weak-form inefficient compared to the NYSE indices. Then we demonstrate through the use of univariate Generalized Hyperbolic subclass of distributions, that a more realistic *VaR* estimate can be obtained. From the *VaR* backtesting results we can conclude that for all the indices, the GHD and the NIG distribution perform much better and provide an accurate estimate of *VaR*. This also justifies the use of sophisticated estimation algorithms used for estimating the GHD and the NIG distribution. The Gaussian distribution gives a reasonable fit for the NYSE indices. On the other hand, for the Indian stock exchanges the assumption of Gaussianity gives results that are too optimistic. Overall, we find that the GHD provides more realistic *VaR* estimates in the univariate case. The *VaR* estimates were found to be twice as much better.

Next, we have tackled the challenging problem of *VaR* estimation of a portfolio comprising mainly of equity assets. Here, copula functions were effectively used to model portfolio risk [30, 48]. Particularly we used the Gaussian and Student's *t* copulas. Here we found that the Student's *t* copula gave a better approximation to portfolio risk.

Lastly, we demonstrated the use of multivariate Generalized Hyperbolic distributions to obtain significantly better estimates of portfolio risk as compared to the Student's *t* copula. We also described an EM type algorithm for calibration of multivariate Generalized Hyperbolic distributions. We highlight the computational challenge involved in calibrating high dimensional Generalized Hyperbolic distributions. Similar to the univariate case, the *VaR* estimates obtained form multivariate Generalized Hyperbolic distributions, particularly the Normal Inverse Gaussian distribution, were found to be twice as better.

Thus we can conclude that the *heavy-tailed* subclass of Generalized Hyperbolic distributions provide a much superior model for risk, especially for Indian markets. This will indeed be of great help in protecting a firm's (bank's) portfolios against adverse market movements.

References

[1] S. K. Sathe, R. K. Lagu, and U. B. Desai, "Investigating Efficiency of the Indian Equities Market with Application to Risk Management," *Indian Chartered Finance Analyst Institute's Journal of Applied Finance*, May 2006.

[2] E. Fama, "Efficient Capital Markets: A Review of Theory and Empirical Work," *The Journal of Finance*, vol. 25, pp. 383–417, May 1970.

[3] O. E. Barndorff-Nielson, "Exponentially Decreasing Distributions for the Logarithm of Particle Size," *Proceedings of the Royal Society London A*, vol. 353, pp. 401–419, 1977.

[4] ——, "Hyperbolic Distributions and Distributions on Hyperbolae," *Scandinavian Journal of Statistics*, vol. 5, pp. 151–157, 1978.

[5] E. Eberlein and U. Keller, "Hyperbolic Distributions in Finance," *Bernoulli*, vol. 1, pp. 281–299, 1995.

[6] K. Prause, "The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures," Ph.D. dissertation, Univ. of Freiburg, Freiburg, October 1999.

[7] A. McNeil, P. Embrechts, and R. Frey, *Quantitative Risk Management: concepts, techniques, and tools.* Princeton University Press, 2005.

[8] D. G. Luenberger, *Investment Science.* New York: Oxford University Press, 1997.

[9] J. Y. Campbell, A. W. Lo, and A. C. MacKinlay, *The Econometrics of Financial Markets.* Princeton, New Jersey: Princeton University Press, 1997.

[10] D. D. Mari and S. Kotz, *Correlation and Dependence.* London: Imperial College Press, 2001.

[11] G. E. P. Box, G. M. Jenkins, and G. C. Reinsel, *Time Series Analysis: Forecasting and Control.* Singapore: Pearson Education, 1994.

[12] A. Dionisio, R. Menezes, and D. A. Mendes, "Mutual information: a dependence measure for nonlinear time series," Economics Working Paper Archive at WUSTL," Working Paper, 2003.

[13] Z. Ding, C. W. Granger, and R. F. Engle, "A long memory property of stock returns and a new model," *Journal of Empirical Finance*, vol. 1, pp. 83–106, Feburary 1993.

[14] S. Poshakwale, "The Random Walk Hypothesis in the Emerging Indian Stock Market," *The Journal of Business Finance and Accounting*, vol. 29, pp. 1275–1299, December 2002.

[15] C. W. Granger, E. Maasoumi, and J. Racine, "A dependence metric for possibly nonlinear processes," *Journal of Time Series Analysis*, vol. 25, pp. 649–669, September 2004.

[16] T. M. Thomas and J. A. Cover, *Elements of Information Theory.* New York: John Wiley and Sons, Inc., 1991.

[17] C. E. Shannon, "A Mathemactical Theory of Communication," *The Bell Systems Technical Journal*, vol. 27, pp. 379–423,623–656, October 1948.

[18] G. A. Darbellay, "The Mutual Information as a Measure of Statistical Dependence," presented at the IEEE International Symposium on Information Theory, 1997.

[19] D. A. Hsieh, "Testing for Nonlinear Dependence in Daily Foreign Exchange Rates," *The Journal of Business*, vol. 62, pp. 339–368, July 1989.

[20] RiskMetrics, J.P. Morgan, "Technical document," 1996.

[21] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions.* New York: Dover Publications, 1968.

[22] E. Bølviken and F. E. Benth, "Quantification of Risk in Norwegian Stocks via the Normal Inverse Gaussian Distribution," in *AFIR Colloquium*, Tromso, Norway, June 2000.

[23] D. Karlis, "An EM type algorithm for maximum likelihood estimation of the Normal-Inverse Gaussian distribution," *Statistics and Probability Letters*, vol. 57, pp. 43–52, March 2002.

[24] O. E. Barndorff-Nielson and R. Stelzer, "Absolute moments of Generalized Hyperbolic Distributions and approximate scaling of Normal-Inverse Gaussian Lévy-processes," Centre for Analytical Finance, University of Aarhus," Working Paper No. 178, 2004.

[25] Y. Chen, W. Hardle, and S. Jeong, "Nonparametric Risk Management with Generalized Hyperbolic Distributions," Centre for Applied Statistics and Economics, Humboldt University, Berlin," Discussion Paper, 2005.

[26] J. C. Lagarias, J. A. Reeds, M. H. Wright, and P. E. Wright, "Convergence properties of the Nelder-Mead Simplex method in low dimensions," *SIAM Journal of Optimization*, vol. 9, pp. 112–147, 1998.

[27] S. K. Sathe, "*Matlab*® implementation of the EM type algorithm for estimating Normal-Inverse Gaussian distribution," http://www.ee.iitb.ac.in/uma/~saket/data/nigem.m, 2005.

[28] P. F. Christoffersen, "Evaluating Interval Forecasts," *International Economic Review*, vol. 39, pp. 841–862, 1998.

[29] P. Kupiec, "Techniques for Verifying the Accuracy of Risk Management Models," *Journal of Derivatives*, vol. 3, pp. 73–84, 1995.

[30] P. Embrechts, F. Lindskog, and A. McNeil, *Modelling Dependence with Copulas and Applications to Risk Management*, S. Rachev, Ed.   Elsevier, 2003.

[31] R. B. Nelsen, *An Introduction to Copulas*.   New York: Springer-Verlag, 1998.

[32] P. Schönbucher, *Credit derivatives pricing models*.   New York: John Wiley & Sons, 2003.

[33] S. S. Galiani, "Copula Functions and their Application in Pricing and Risk Managing Multiname Credit Derivative Products," Ph.D. dissertation, Kings College London, London, September 2003.

[34] C. Kimberling, "A probabilistic interpretation of complete monotonicity," *Aequationes Mathematicae*, vol. 10, no. 2, pp. 152–164, 1974.

[35] A. Marshall and I. Olkin, "Families of Multivariate Distributions," *Journal of the American Statistical Association*, vol. 83, no. 403, pp. 834–841, 1988.

[36] E. Frees and E. Valdez, "Understanding relationships using copulas," *North American Actuarial Journal*, vol. 2, no. 1, pp. 1–25, 1998.

[37] R. Mashal and M. Naldi, "Pricing Multiname Credit Derivatives: Heavy Tailed Hybrid Approach," *Preprint*, 2002.

[38] R. Mashal and A. Zeevi, "Beyond Correlation: Extreme Co-movements Between Financial Assets," *Manuscript, Columbia University*, 2002.

[39] H. Joe and J. Xu, "The estimation method of inference functions for margins for multivariate models," Department of Statistics, University of British Columbia, Tech. Rep. 166, 1996.

[40] F. Lindskog, "Linear correlation estimation," *Preprint, ETH Zurich*, 2000.

[41] F. Lindskog, A. McNeil, and U. Schmock, "Kendall?s tau for elliptical distributions," *Preprint, ETH Zurich*, 2001.

[42] P. Jäckel, *Monte Carlo methods in Finance.* New York: John Wiley and Sons, Inc., 2003.

[43] P. Glasserman, *Monte Carlo Methods in Financial Engineering.* Springer, 2003.

[44] P. Blaesild, "The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data," *Biometrika*, vol. 68, no. 1, p. 251, 1981.

[45] K. Aas and I. Haff, "The Generalized Hyperbolic Skew Student's t-Distribution," *Journal of Financial Econometrics*, 2006.

[46] R. Protassov, "EM-based maximum likelihood parameter estimation for multivariate generalized hyperbolic distributions with fixed λ," *Statistics and Computing*, vol. 14, no. 1, pp. 67–77, 2004.

[47] J. Mencıa and E. Sentana, "Estimation and testing of dynamic models with Generalised Hyperbolic innovations," Mimeo, CEMFI, Tech. Rep., 2003.

[48] R. Frey, A. J. McNeil, and M. Nyfeler, "Copulas and credit models," *RISK*, pp. 111–114, October 2001.

[49] G. A. Darbellay and D. Wuertz, "The entropy as a tool for analyzing statistical dependences in financial time series," *Physica A: Statistical Mechanics and its Applications*, vol. 287, pp. 429–439, December 2000.

Appendix A

Long memory property of stock returns

It is a well understood fact that autocorrelations of squares or absolute values of returns show some kind of "long-range dependence" (Ding *et al.* [13]). Except for very short time lags, linear autocorrelations are not statistically different from zero. However, they are insufficient in making riskless profit.

Furthermore, if $\{x_t\}$ is *i.i.d* then any transformation of $\{x_t\}$ is also *i.i.d*. Thus, along with $\{x_t\}$,$\{|x_t|\}$ and $\{x_t^2\}$ are also *i.i.d*. But empirical analysis shows that sample autocorrelation coefficients of $\{|x_t|\}$ and $\{x_t^2\}$ are significant and positive over long lags. This points out two things, first, the underlying process is not *i.i.d*, and second, linear autocorrelations are not a good measure for studying dependence.

On the other hand mutual information is invariant under bijective transformations (Darbellay and Wuertz [49]), as a result,

$$I(|x_t|^q, |x_{t+h}|^q) = I(|x_t|, |x_{t+h}|) \text{ for all } q > 0, \tag{A.1}$$
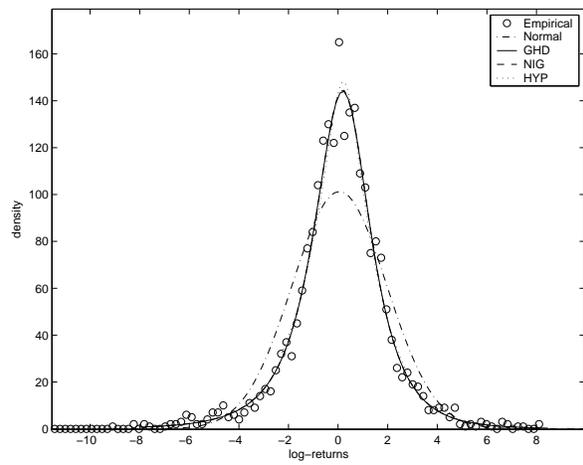
Thus, except for the first few lags, all the information about the present returns contained in some past returns is carried by the amplitude of the returns and not by their signs. The same conclusion was verified for NSE stock indices returns series, except for small differences between $I(x_t, x_{t+h})$ and $I(|x_t|, |x_{t+h}|)$ for the first few days.
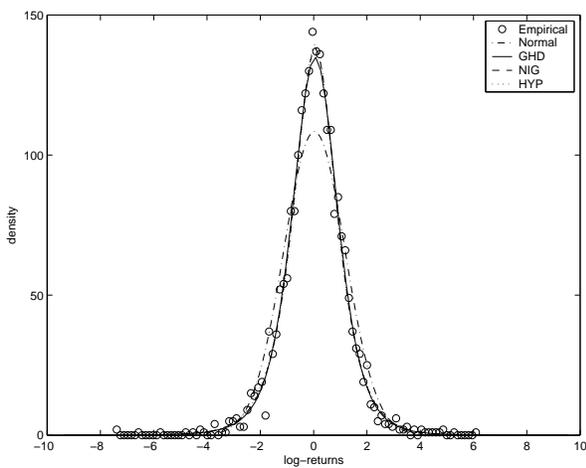
Appendix B

Tables and Figures
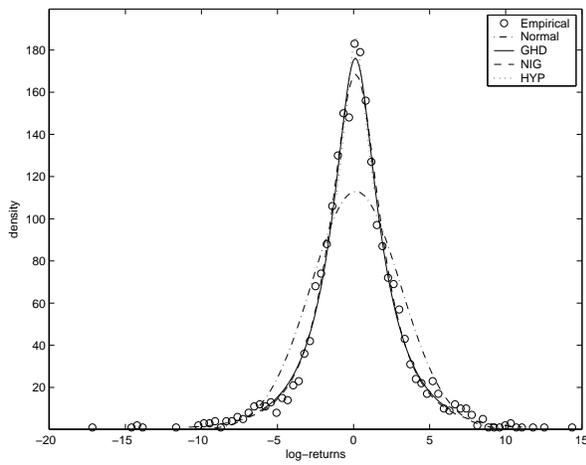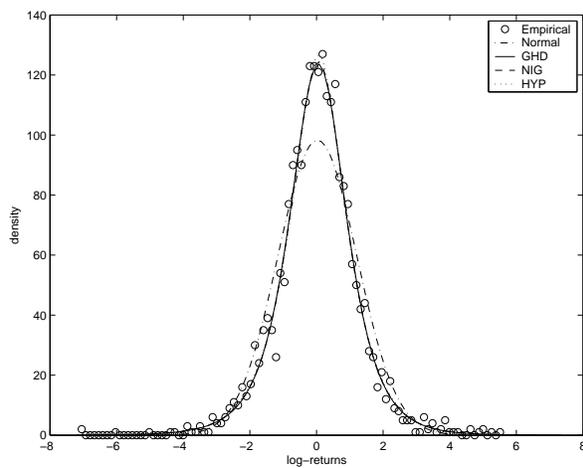

(a) CNXIT   (b) NIFJUN

Figure B.1: Density functions of NSE stock indices returns series.
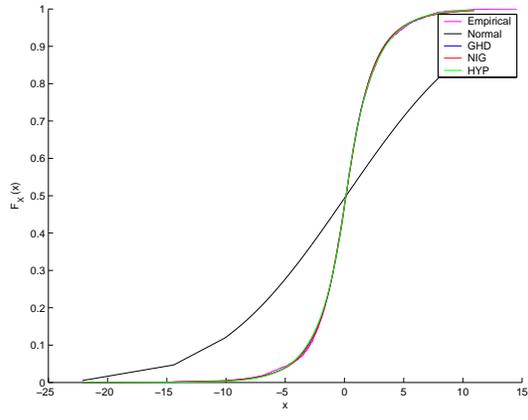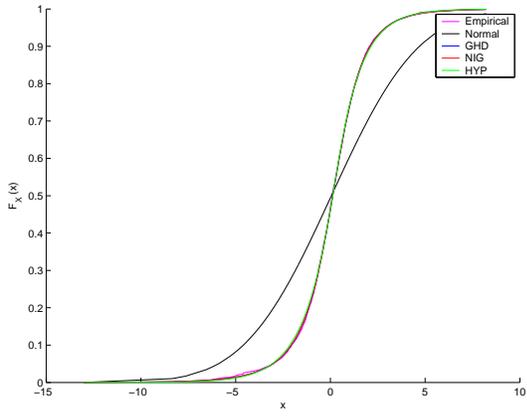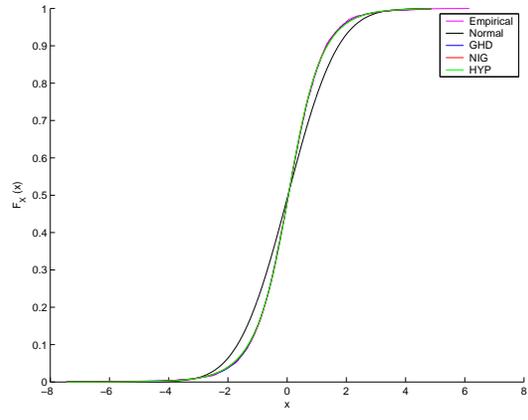

(a) DOWJI   (b) S&P500

Figure B.2: Density functions of NYSE stock indices returns series.
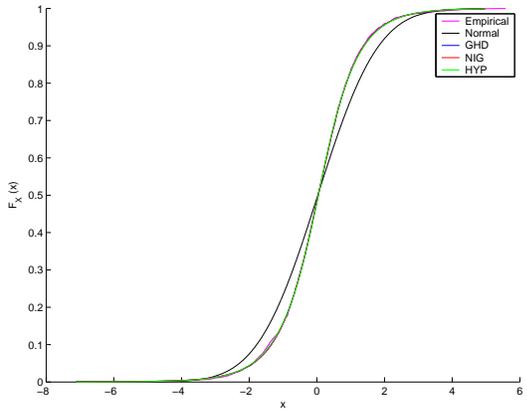
(a) CNXIT

(b) NIFJUN

(c) DOWJI

(d) S&P500

Figure B.3: Distribution functions of NSE and NYSE stock indices returns series.

Table B.1: BDS test for NIFTY, NIFJUN and CNX500.

N	$l/\sigma = 0.5$	$l/\sigma = 0.75$	$l/\sigma = 1$	$l/\sigma = 1.25$	$l/\sigma = 1.5$
NIFTY					
2	8.3247	8.5667	9.0648	9.4208	9.9135
3	11.1325	10.9873	11.2979	11.4196	11.7266
4	13.1133	12.7460	12.9677	12.9084	13.0749
5	14.9405	13.9236	13.9201	13.6242	13.6164
6	17.7145	15.8368	15.4755	14.8422	14.5368
7	21.3451	18.4323	17.6003	16.3923	15.6756
8	26.5106	21.4488	19.8925	17.9719	16.7672
9	32.0704	24.7786	22.4647	19.5995	17.8061
10	43.1343	29.1762	25.3348	21.3548	18.8307
11	60.4570	34.6616	28.8460	23.3864	20.0141
12	96.4635	42.3725	33.1773	25.6881	21.2940
13	172.2944	52.6610	38.4053	28.3292	22.7196
14	313.3938	67.1612	44.8438	31.5907	24.4459
15	485.4985	86.0596	52.7567	35.3725	26.3416
16	770.8998	111.3477	62.1980	39.6756	28.4186
NIFJUN					
2	12.9600	13.6844	14.5038	15.1360	15.5903
3	16.3196	17.0740	17.8407	18.3658	18.6496
4	18.8119	19.3420	19.9224	20.2627	20.3031
5	21.1401	21.2837	21.4770	21.4549	21.1792
6	25.1727	24.4856	23.9494	23.2865	22.4700
7	30.1629	28.4475	26.9601	25.4917	23.9926
8	36.5364	33.0590	30.2932	27.7633	25.5154
9	44.3526	38.8262	34.2067	30.3324	27.1487
10	56.9503	46.3739	39.1610	33.3735	28.9709
11	77.3130	55.6157	44.8945	36.8258	30.9965
12	107.0872	67.4043	51.6962	40.6671	33.1888
13	160.1281	82.2372	59.5020	45.0175	35.5980
14	237.2529	102.7292	69.3346	50.1930	38.3838
15	337.6042	128.7782	81.6898	56.3240	41.5574
16	532.9275	162.4183	96.6214	63.3706	45.0482
CNX500					
2	10.9108	11.6772	12.3008	13.1298	13.5824
3	12.8622	13.8190	14.6656	15.5800	15.9384
4	14.0713	15.1508	16.0096	16.9701	17.2096
5	15.9247	16.9714	17.4605	18.1646	18.1725
6	18.8792	19.2289	19.1961	19.4119	19.0076
7	22.4515	22.3901	21.6848	21.2271	20.1988
8	27.1248	26.1527	24.5196	23.2095	21.4744
9	34.1614	31.1529	27.8795	25.4040	22.7716
10	44.0512	38.2220	32.2645	28.0671	24.2399
11	51.1138	47.0494	37.5957	31.2165	25.9228
12	56.7556	57.5270	43.8293	34.7377	27.7419
13	53.0310	71.3963	51.3039	38.8442	29.7617
14	53.3702	89.3202	60.7306	43.8166	32.1256
15	56.2968	113.4075	72.4910	49.8093	34.8514
16	75.0361	146.5072	86.8372	57.0512	38.0098

Table B.2: BDS test for CNXMID, CNXIT and NIFBNK.

N	$l/\sigma = 0.5$	$l/\sigma = 0.75$	$l/\sigma = 1$	$l/\sigma = 1.25$	$l/\sigma = 1.5$
CNXMID					
2	12.5168	12.8780	13.1204	13.2749	13.3075
3	14.6591	15.3566	15.2797	15.3015	15.1560
4	16.7295	17.2646	16.6957	16.5224	16.2525
5	19.4767	19.2878	18.0443	17.4466	16.9993
6	23.9922	22.4178	19.9936	18.6823	17.8453
7	30.5468	26.3545	22.2655	20.0327	18.6915
8	41.1024	31.4062	24.8991	21.4764	19.5297
9	54.6667	37.8173	28.1775	23.3210	20.5884
10	74.1967	46.7469	32.3086	25.4626	21.7477
11	109.9485	58.5297	37.3474	27.9859	23.0598
12	178.6524	74.9716	43.6124	30.8093	24.4662
13	316.5064	97.7427	51.3231	33.9597	25.9146
14	559.1235	129.3425	61.1942	37.8165	27.6410
15	1075.3295	174.0655	74.0466	42.5655	29.6721
16	2027.8826	235.8878	89.7458	48.1494	31.9828
CNXIT					
2	17.4613	17.7551	17.8638	17.7460	17.4923
3	21.6687	21.5601	21.3276	20.8442	20.0401
4	25.6859	24.6256	23.7648	22.8308	21.5963
5	30.5436	28.1235	26.1751	24.6340	22.9513
6	38.3379	33.1111	29.2926	26.7273	24.3951
7	51.1551	40.3457	33.6617	29.5145	26.2067
8	67.9234	49.2140	38.7807	32.6840	28.2425
9	90.3582	60.7601	44.8271	36.1741	30.3822
10	121.2481	76.9143	52.8197	40.4417	32.8025
11	166.8313	99.4265	63.0666	45.5138	35.4933
12	225.2739	128.4258	75.4571	51.2383	38.3723
13	295.1724	168.0127	91.1228	57.8338	41.5220
14	372.7417	222.5972	110.6229	65.4499	45.0080
15	456.5997	296.2385	135.6671	74.5427	48.9748
16	518.2548	403.3797	169.0413	85.9084	53.7273
NIFBNK					
2	7.3048	7.5540	7.7690	8.0845	8.6433
3	8.2505	8.3926	8.3581	8.4352	8.7734
4	10.5132	10.1385	9.8475	9.6416	9.7440
5	12.0635	11.4401	10.8224	10.3204	10.1627
6	13.5462	12.5597	11.7110	10.9114	10.4865
7	15.7259	14.1757	12.7416	11.5971	10.9194
8	21.0577	16.8053	14.2130	12.4964	11.4278
9	26.6883	20.3009	15.9800	13.5309	12.0254
10	33.4284	24.5746	18.1351	14.7837	12.7343
11	41.9437	30.3130	20.6643	16.2540	13.5513
12	43.7911	36.7395	23.4981	17.8773	14.4029
13	51.0626	46.9370	27.5004	19.8321	15.3480
14	70.1490	59.0817	32.2625	22.0552	16.3570
15	63.5493	78.2522	38.1559	24.5215	17.3692
16	-1.8751	98.3284	46.0102	27.5728	18.6094

Table B.3: BDS test for DOWJI and S&P500.

N	$l/\sigma = 0.5$	$l/\sigma = 0.75$	$l/\sigma = 1$	$l/\sigma = 1.25$	$l/\sigma = 1.5$
DOWJI					
2	3.3972	4.1565	4.7501	5.5580	6.4425
3	6.2482	7.2115	7.7278	8.5295	9.3802
4	7.5415	8.5843	9.2103	10.1315	10.9964
5	9.3388	10.2474	10.8719	11.7533	12.5349
6	11.4420	12.2139	12.6311	13.3557	13.9691
7	13.6170	14.3394	14.4163	14.8451	15.1556
8	15.5277	17.1853	16.8944	16.7407	16.5842
9	18.2212	20.4204	19.6525	18.8139	18.1347
10	19.4773	24.1288	22.8730	21.1587	19.8036
11	20.0666	27.6664	26.2851	23.6230	21.5175
12	24.1527	33.1897	30.7380	26.5983	23.4923
13	38.0007	39.6249	35.9369	30.0354	25.6968
14	54.5134	47.7770	42.2707	33.9873	28.1270
15	37.3805	59.3374	50.2293	38.7167	30.8492
16	-2.5348	76.0235	59.9221	44.3302	33.9297
S&P500					
2	4.2084	4.3767	4.8549	5.5675	6.3903
3	7.6722	7.9140	8.2545	8.8817	9.5248
4	9.5209	9.9710	10.2293	10.8259	11.4104
5	12.3163	12.6440	12.7077	13.1230	13.5232
6	15.3324	15.4615	15.1703	15.2558	15.3189
7	18.9351	18.5816	17.6619	17.2337	16.8630
8	24.2205	22.5398	20.6980	19.5140	18.5167
9	29.6038	27.1840	24.1937	22.0901	20.3203
10	35.4974	32.5784	28.1959	24.9565	22.2476
11	43.6156	39.2007	32.8121	28.1396	24.3149
12	58.0798	47.8628	38.3565	31.8820	26.6455
13	60.6955	57.5638	44.9012	36.2287	29.2441
14	71.2939	69.5370	53.0388	41.2472	32.1343
15	38.9129	85.2866	62.6254	47.0702	35.3820
16	-2.3969	107.1867	74.2323	54.0498	39.0636